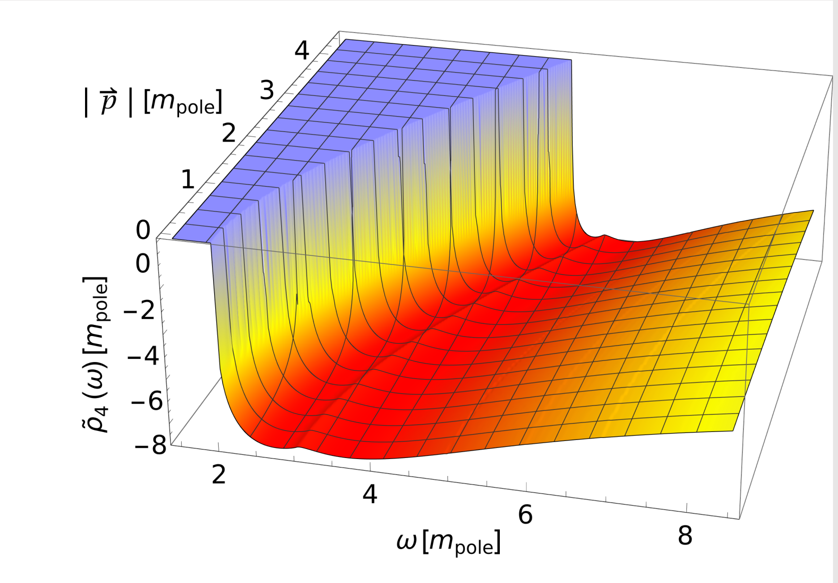
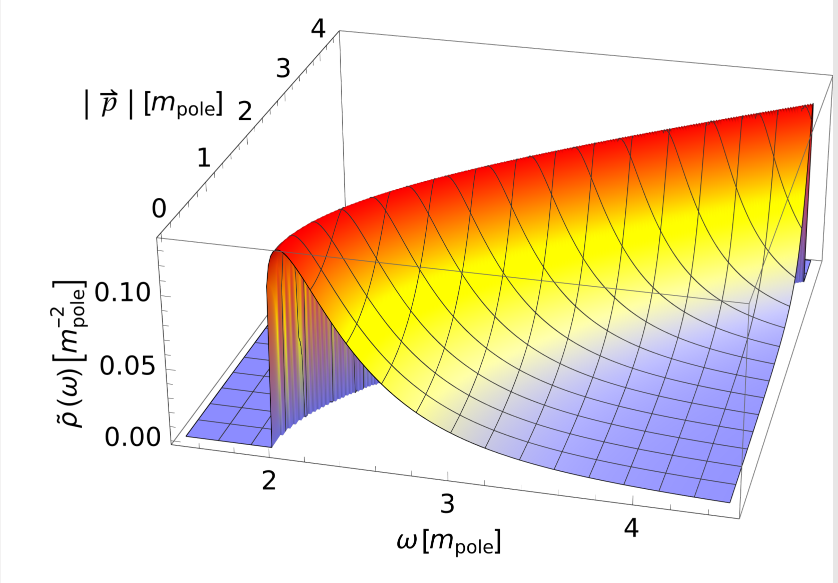
Spectral functions from spectral flows

Jonas Wessely

Theoretical hadron physics Lunch Club Seminar

Giessen 31.05.23





In collaboration with **Jan Horak, Friederike Ihssen, Jan M. Pawlowski, Nicolas Wink**

- arXiv:2303.16719

Outline

- Real time correlators with *spectral* functional methods

  → **(Heavy) Quark diffusion**

- Spectral fRG and the Callan-Symanzik cut-off

- Results for real scalar fields in (2+1) dimensions

Real time correlators with spectral functional methods

$$\mathscr{D}_s = \lim_{\omega\to 0} \frac{\sigma(\omega, \vec{p} = 0)}{\omega \chi_q \pi}$$

$$\sigma(\omega, \mathbf{p}) = \frac{1}{\pi} \int \mathrm{d}t \, e^{\mathrm{i}\omega t} \int \mathrm{d}^3x \, e^{\mathrm{i}\mathbf{x}\mathbf{p}} \langle [J_i(t, \mathbf{x}), J_i(0,0)] \rangle$$

- Dynamic observables like transport coefficient require real time correlation functions
- Large uncertainties on the lattice

Real time correlators with spectral functional methods

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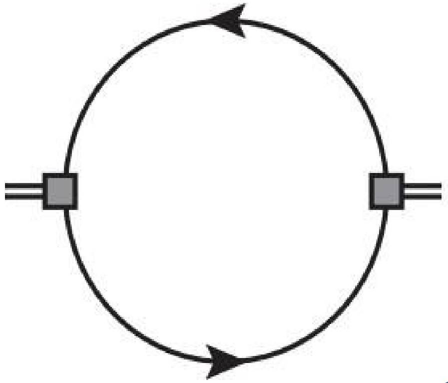
- Dynamic observables like transport coefficient require real time correlation functions
- Large uncertainties on the lattice
- Functional methods: exact diagrammatic expression

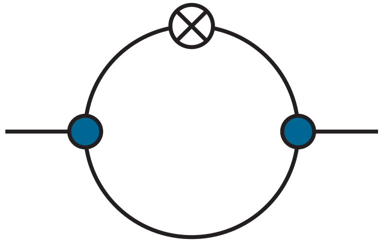
$\sigma(\omega, \mathbf{p}) \propto \mathrm{Im}$ 

Need quark propagator in real time

Spectral functional equations

Spectral diagrams and spectral renormalisation

(Horak, Pawlowski, Wink arXiv: 2006.09778)



$$\propto \int_q G(q)^2 G(p+q)$$

Spectral functional equations

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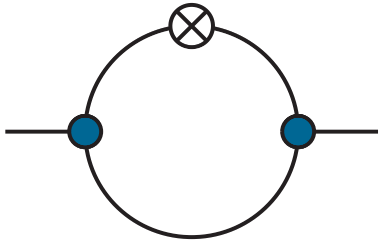

$$\propto \int_q G(q)^2 G(p+q)$$

$$G(p) = \int_\lambda \frac{\rho(\lambda)}{p^2 + \lambda^2}$$

$$= \int_{\lambda_1,\lambda_2,\lambda_3} \rho(\lambda_1)\rho(\lambda_2)\rho(\lambda_2) \int_q \frac{1}{(q^2+\lambda_1^2)(q^2+\lambda_2)((p+q)^2+\lambda_3^2)}$$

Spectral functional equations

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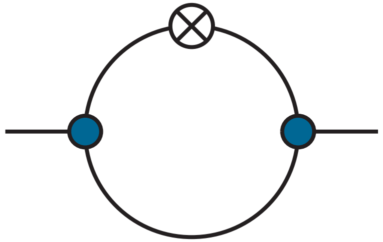

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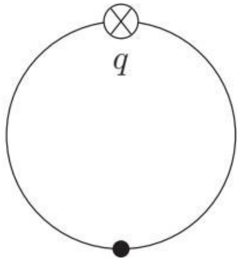
- Loop integrals can be calculated in dimReg
- Access to the full complex plane

- But: additional spectral integrals
- Spectral renormalisation for diverging diagrams

Spectral fRG and the Callan-Symanzik cut-off

arXiv:2206.10232

$$S_\varphi[\phi] \to S_\varphi[\phi] + \frac{1}{2}\int_p \phi(p) R_k^\phi(p)\, \phi(-p) \quad \Longrightarrow \quad \partial_t \Gamma_k = \frac{1}{2}$$



We use the CS cut-off

$$R_k = Z_\phi k^2 \quad \Longrightarrow \quad S^{(2)}|_{\phi=0} = p^2 + m_0 + k^2 = p^2 + m_k^2$$

In Contradistinction to momentum-shell-RG:

A solution to the flow-equation represents a physical theory even for finite k

Spectral fRG and the Callan-Symanzik cut-off

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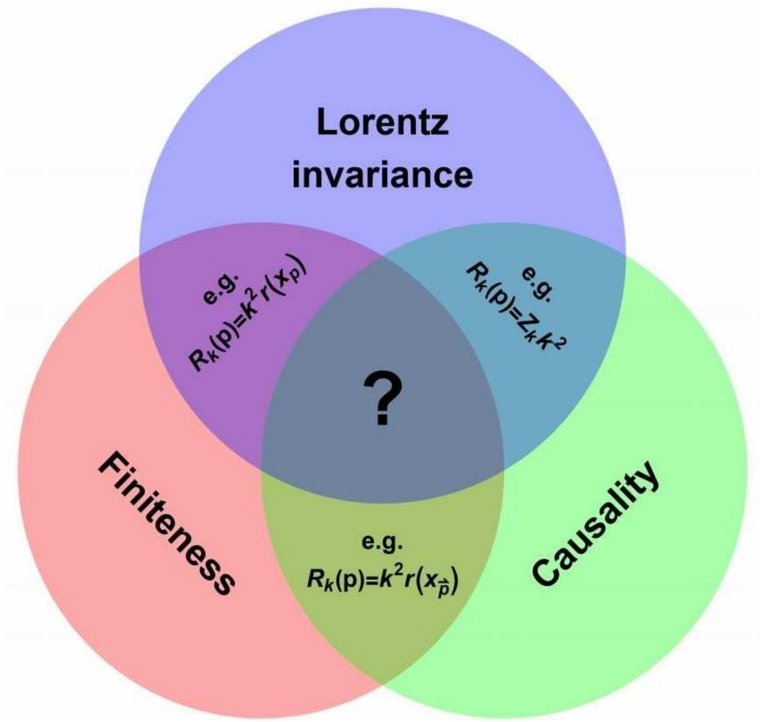
$$S[\phi] \to S[\phi] + \frac{1}{2}\int_q \phi(q) R_k(q^2) \phi(-q) \quad \Longrightarrow \quad G_{(}p) = \frac{1}{\Gamma_k^2(p^2) + R_k(p^2)}$$



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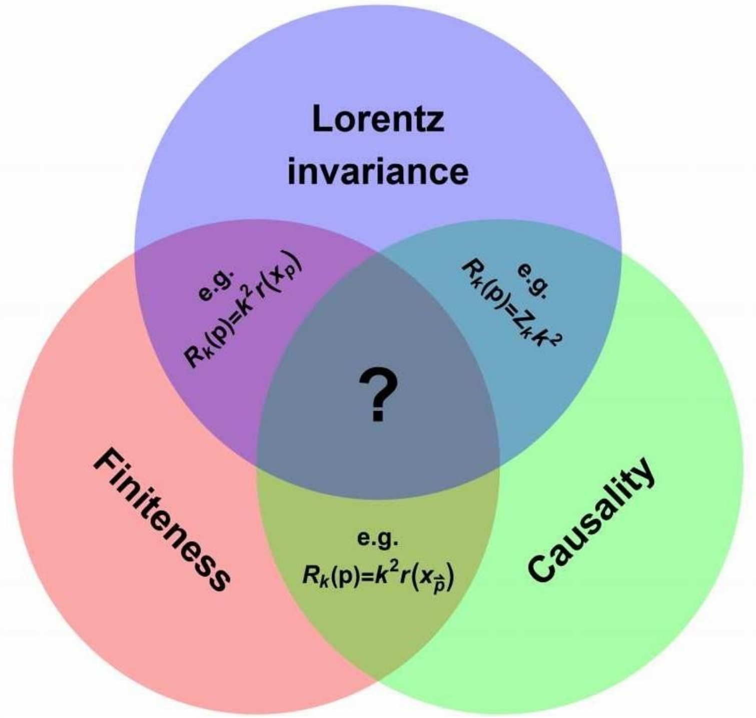
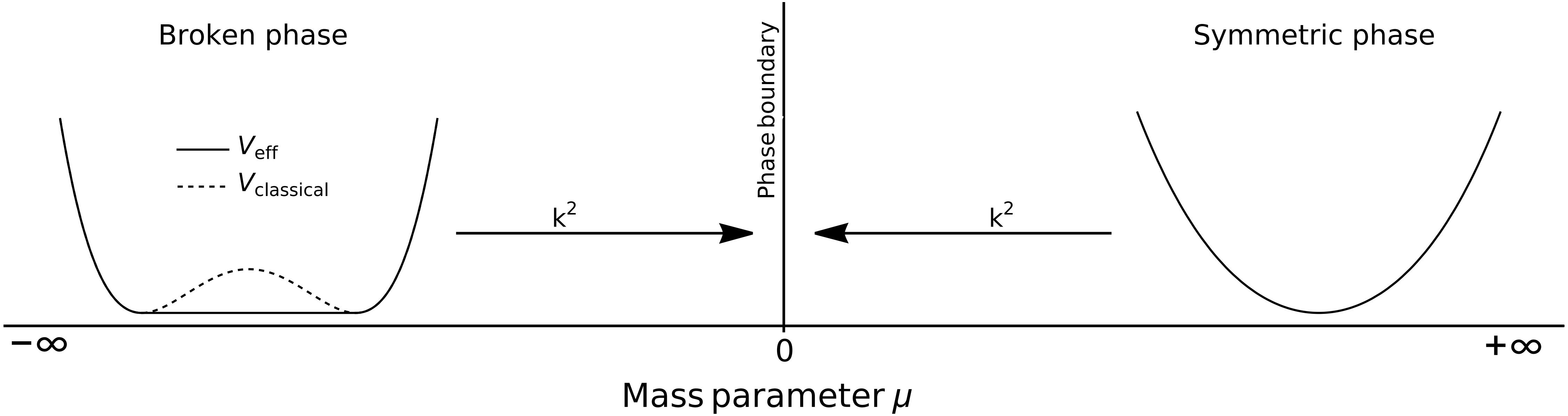


- Have to choose 2 out of 3 properties:
  - UV-regularisation
  - Lorentz invariance
  - Causal propagator at finite k

Spectral fRG and the Callan-Symanzik Cutoff

The flowing mass parameter

$$S[\phi] = \int \mathrm{d}^3x \left\{ \frac{1}{2}\phi\Big(-\partial^2 + \mu\Big)\phi + \frac{\lambda_\phi}{4!}\phi^4 \right\}$$



Spectral fRG and the Callan-Symanzik Cutoff

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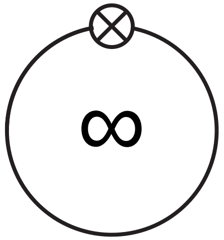
$$\mu\partial_\mu \, \Gamma[\phi] \; = \; \frac{1}{2}$$ 

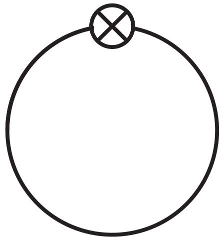
$$+ \; \frac{1}{2}\phi^2$$ 

- Without UV-regularisation, divergent diagram!

Spectral fRG and the Callan-Symanzik Cutoff

The flowing mass parameter

$$S[\phi] = \int \mathrm{d}^3x \left\{ \frac{1}{2}\phi\Big(-\partial^2 + Z_\phi \mu\Big)\phi + \frac{\lambda_\phi}{4!}\phi^4 \right\}$$

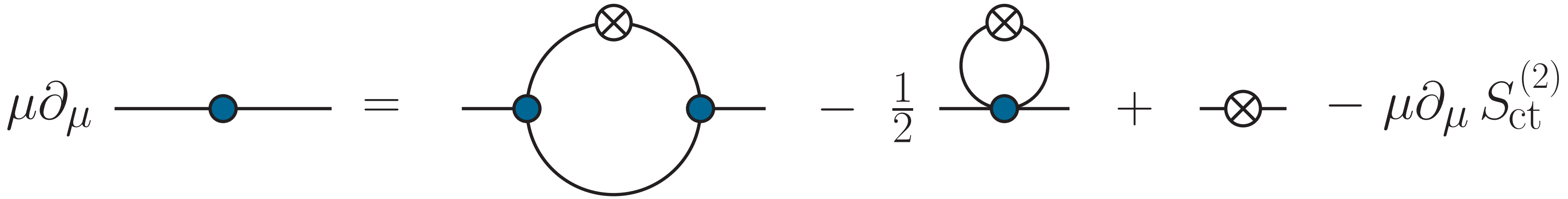
$\mu\partial_\mu \, \Gamma[\phi] \; = \; \frac{1}{2}$  $+ \; \frac{1}{2}\,\phi^2 -\!\otimes\!- \; - \; \mu\partial_\mu \, S_{\mathrm{ct}}[\phi]$

- Without UV-regularisation divergent diagram!
- Introduce counter-term flow via limiting prozedure over UV-finite regulators
- Counter-term flow determined by flowing renormalisation condition

fQCD collaboration: arXiv:2206.10232

Spectral fRG and the Callan-Symanzik cut-off

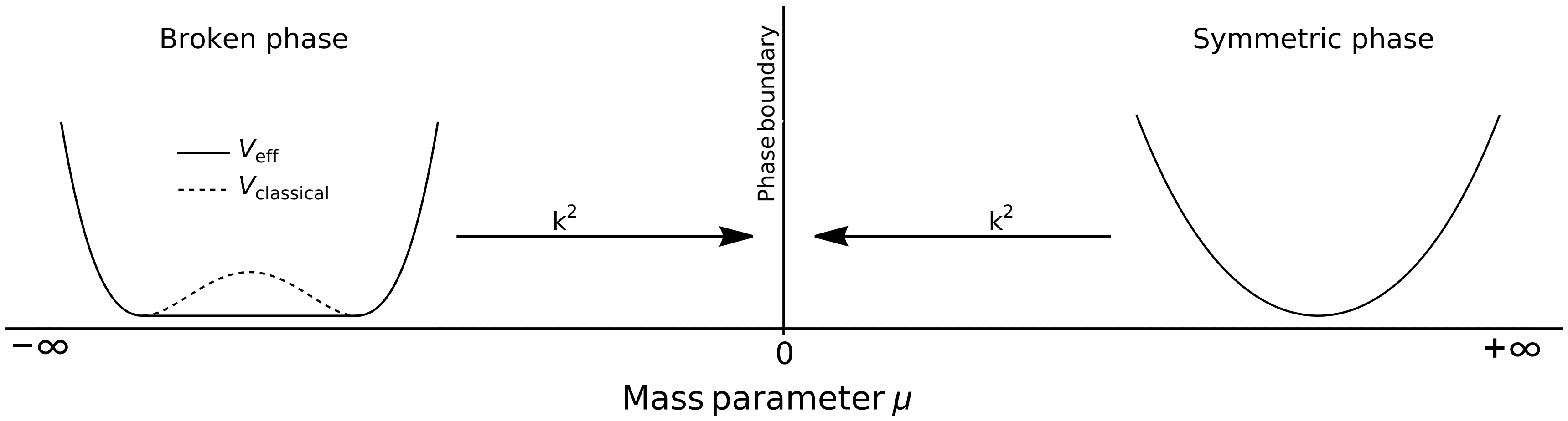
flowing renormalisation conditions



- Diagrams in the flow are finite in (2+1) dimensions since the insertion of the cut-off lowers the degree of divergence by 2
- But: initial condition *implicitly* sets a renormalisation condition
- Exploiting the counter-term gives us the opportunity to control the flow in theory space and eliminates fine-tuning

Real scalar field in 3 dimension

flowing on-shell renormalisation



Classical pole mass on the physical minimum ϕ_0

$$m^2_{\text{pole}} = 2|\mu| = 2k^2$$

Flowing on-shell condition in the broken phase

$$\Gamma^{(2)}[\phi_0]\Big|_{p^2=-2k^2} = 0$$

Classical pole mass at $\phi_0 = 0$

$$m^2_{\text{pole}} = \mu = k^2$$

Flowing on-shell condition in the symmetric phase

$$\Gamma^{(2)}[\phi_0 = 0]\Big|_{p^2=-k^2} = 0$$

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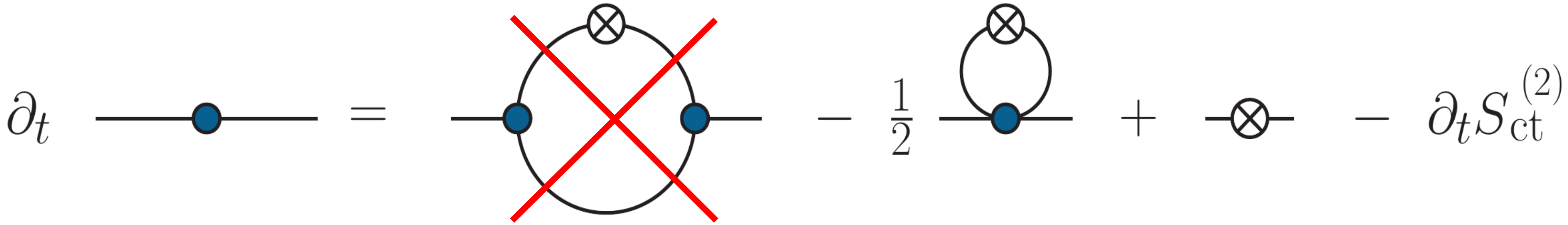
Real scalar field in 3 dimension

Symmetric Phase



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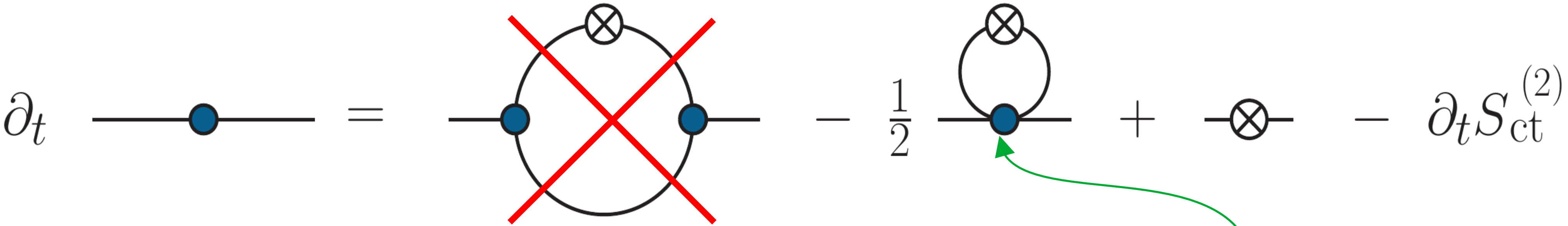
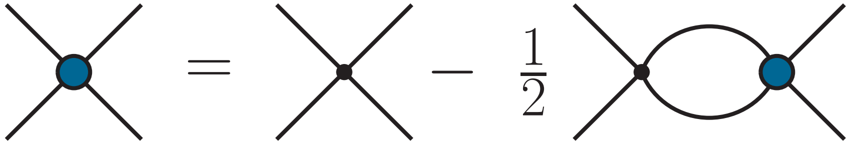


s-channel Bubble Resumation:



$$\Gamma^{(4)}(p^2) = \lambda_\phi + \int_\lambda \frac{\rho_4(\lambda)}{\lambda^2 + p^2}$$

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Real scalar field in 3 dimension

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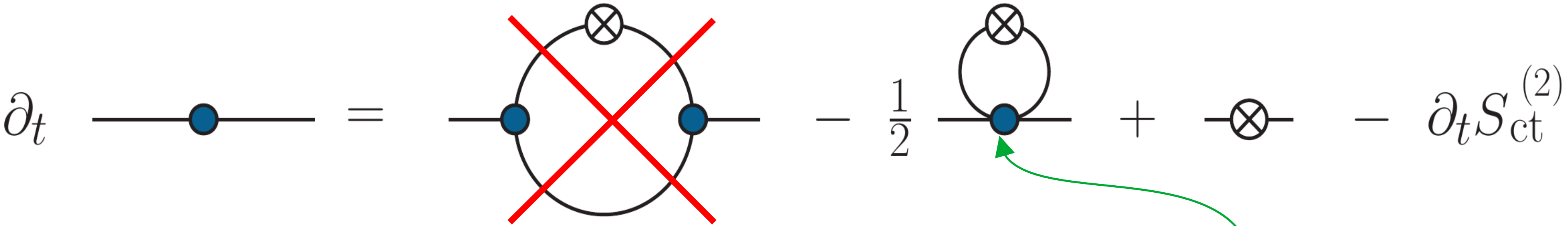

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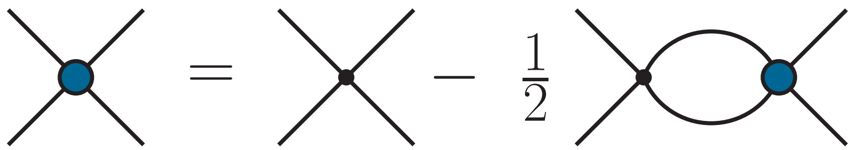

$$\Gamma^{(4)}(p^2) = \lambda_\phi + \int_\lambda \frac{\rho_4(\lambda)}{\lambda^2 + p^2}$$

Strategy:

Integrate the flow of the two-point function

$$\rightarrow \; \Gamma^{(2)}(p^2)$$

Extract spectral function in every step via

$$\rho(\omega) = \frac{-2\mathrm{Im}\,\Gamma^{(2)}}{||\Gamma^{(2)}||^2}\Bigg|_{p^2=-(w+i\,0^+)^2}$$

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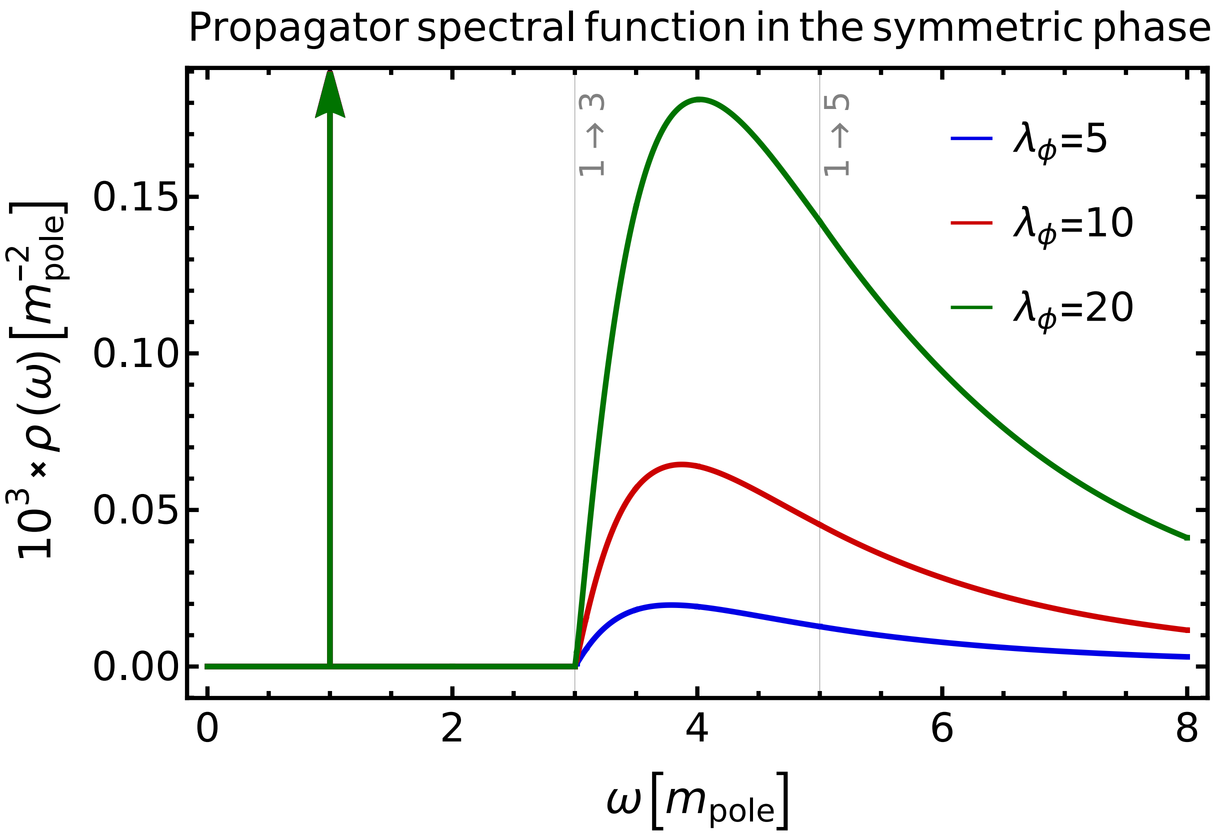
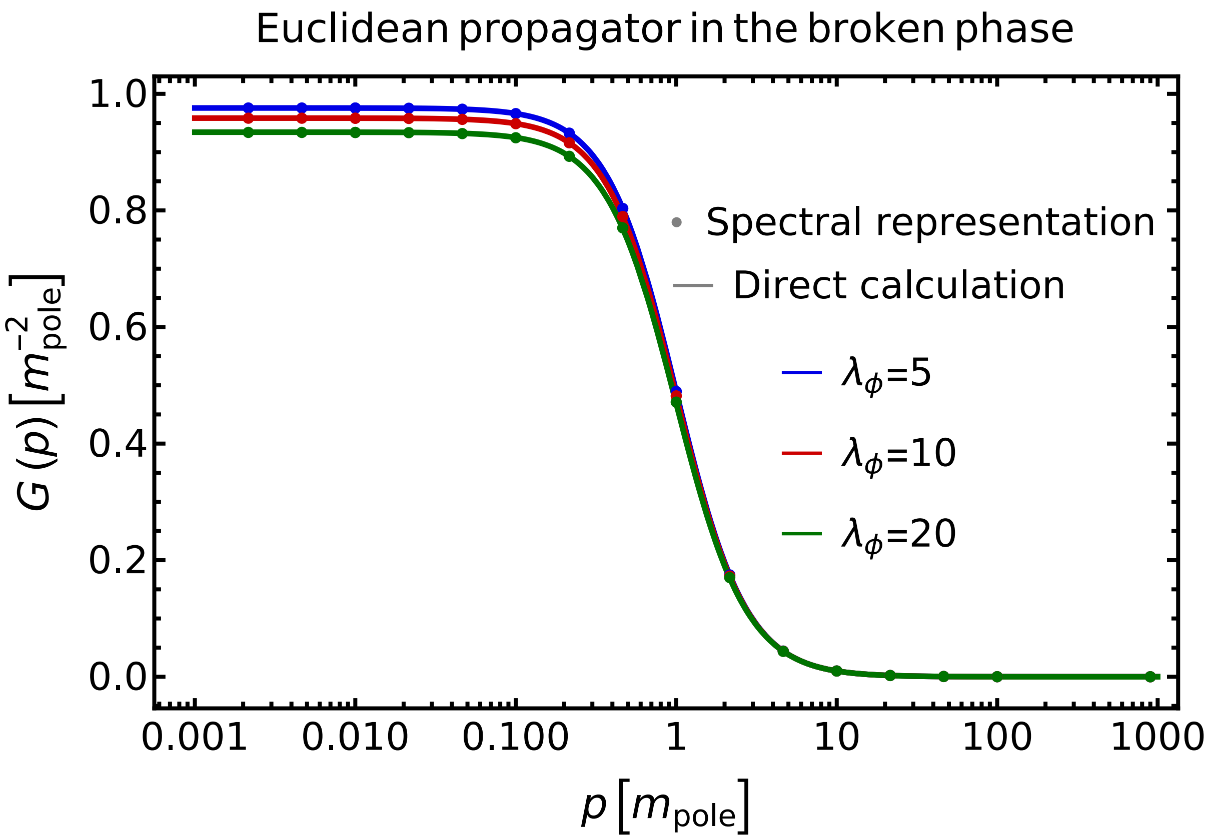
Results in the (symmetric) phase





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Real scalar field in 3 dimension

Broken phase

$$\phi_0^2 \approx \frac{6k^2}{\lambda_\phi}$$

Can't neglect running of the physical minimum!

Flow of the full inverse propagator:

Real scalar field in 3 dimension

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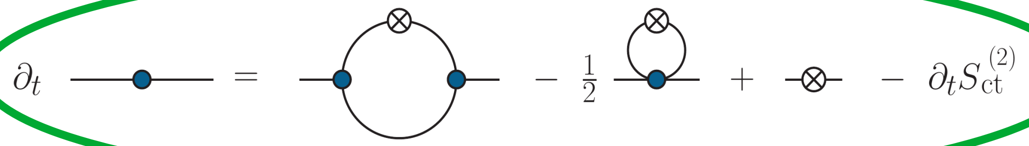
$$\frac{d}{dt}\Gamma^{(2)}[\phi_0](p) = \partial_t\Gamma^{(2)}[\phi_0](p) + \left(\partial_t\phi_0\,\Gamma^{(3)}[\phi_0]\right)(p)$$



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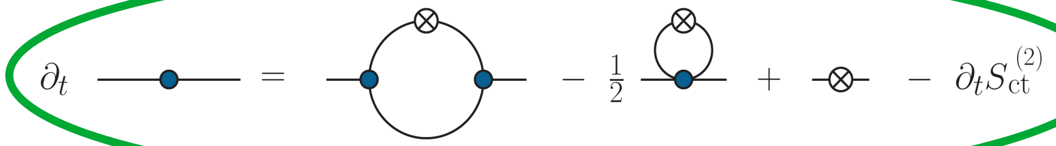

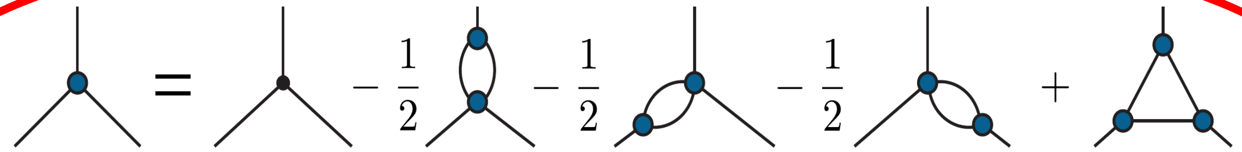

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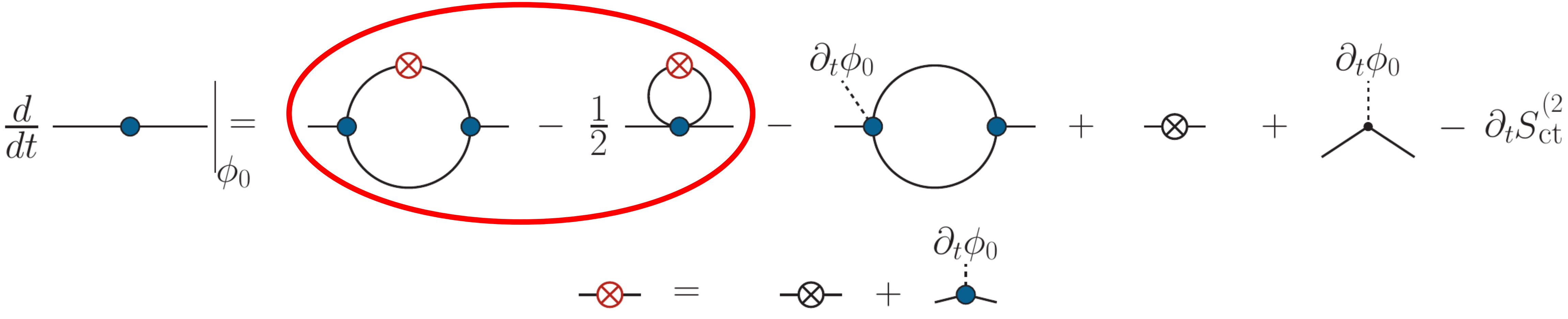
Real scalar field in 3 dimension

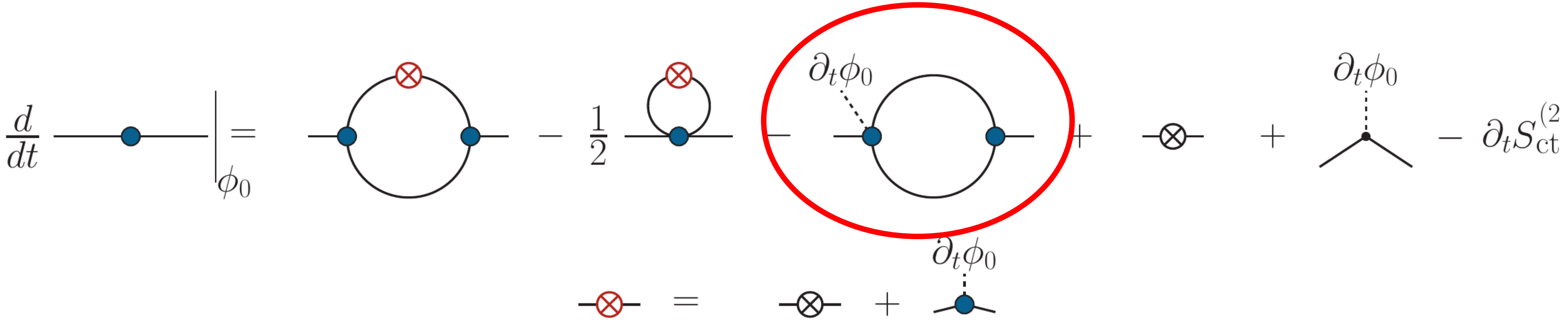
Broken phase



- Generalised Regulator derivative comprises scale-dependence of internal lines

Real scalar field in 3 dimension

Broken phase



- Generalised Regulator derivative comprises scale-dependence of internal lines
- Additional Fish-diagram accounts for the running of the 3-Vertex

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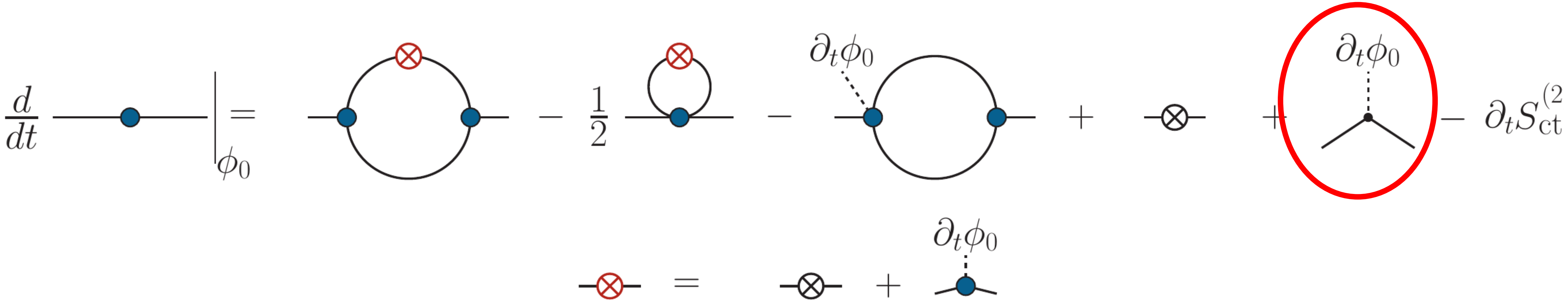
Real scalar field in 3 dimension

Broken phase



- Generalised Regulator derivative comprises scale-dependence of internal lines
- Additional Fish-diagram accounts for the running of the 3-Vertex
- Classical 3-point function necessary for a positive mass-flow!

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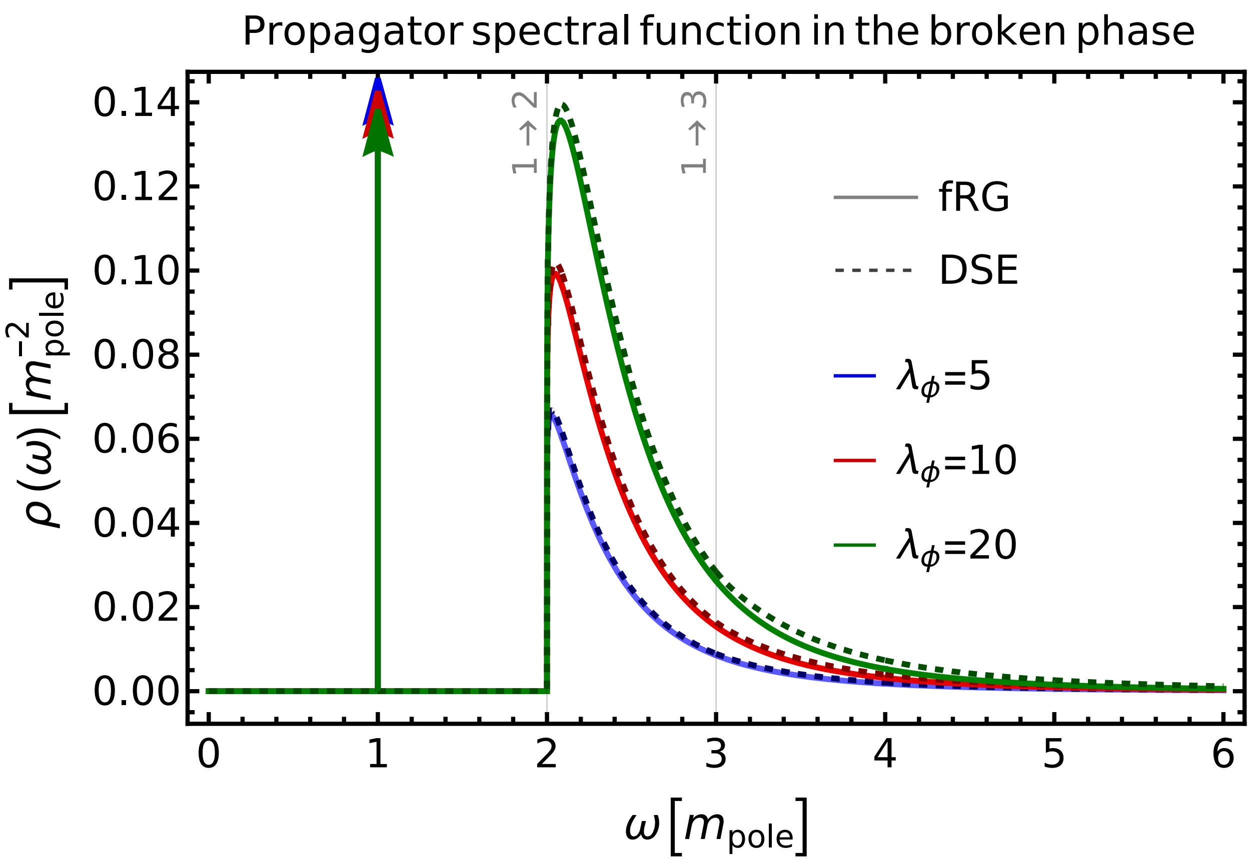
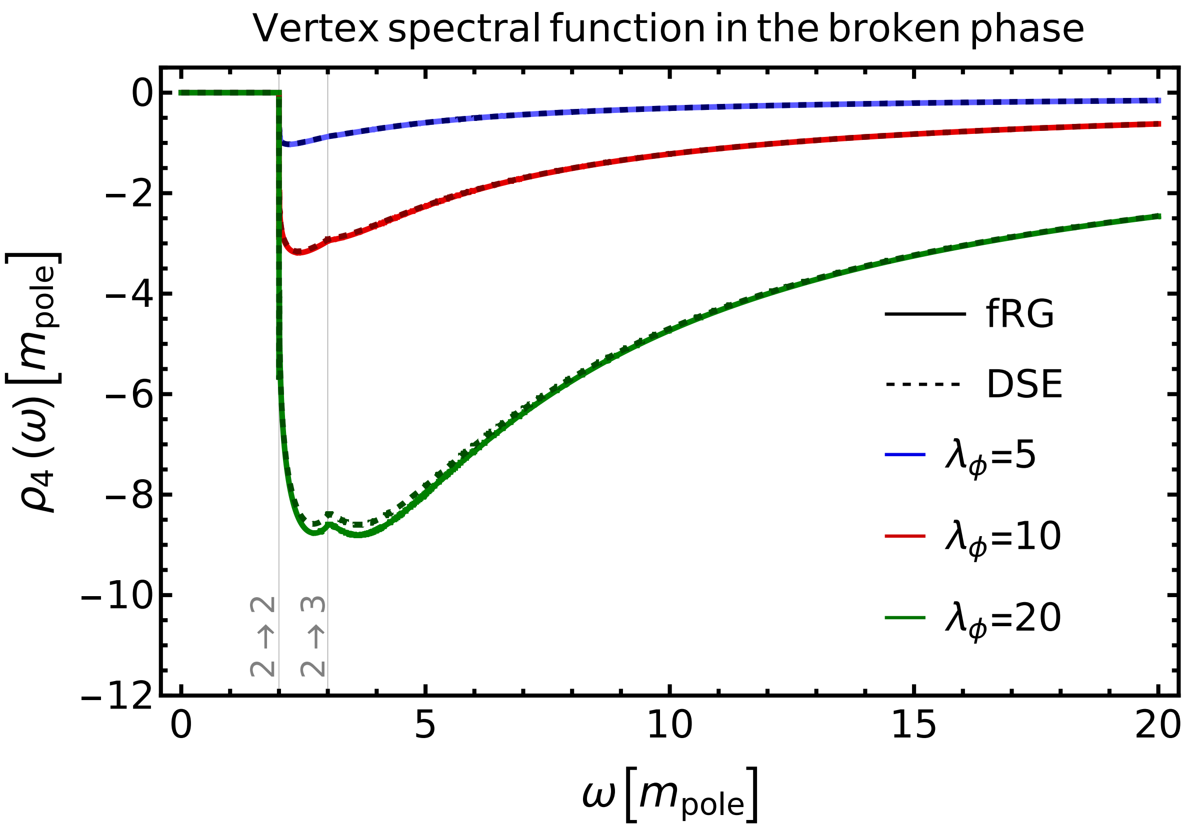
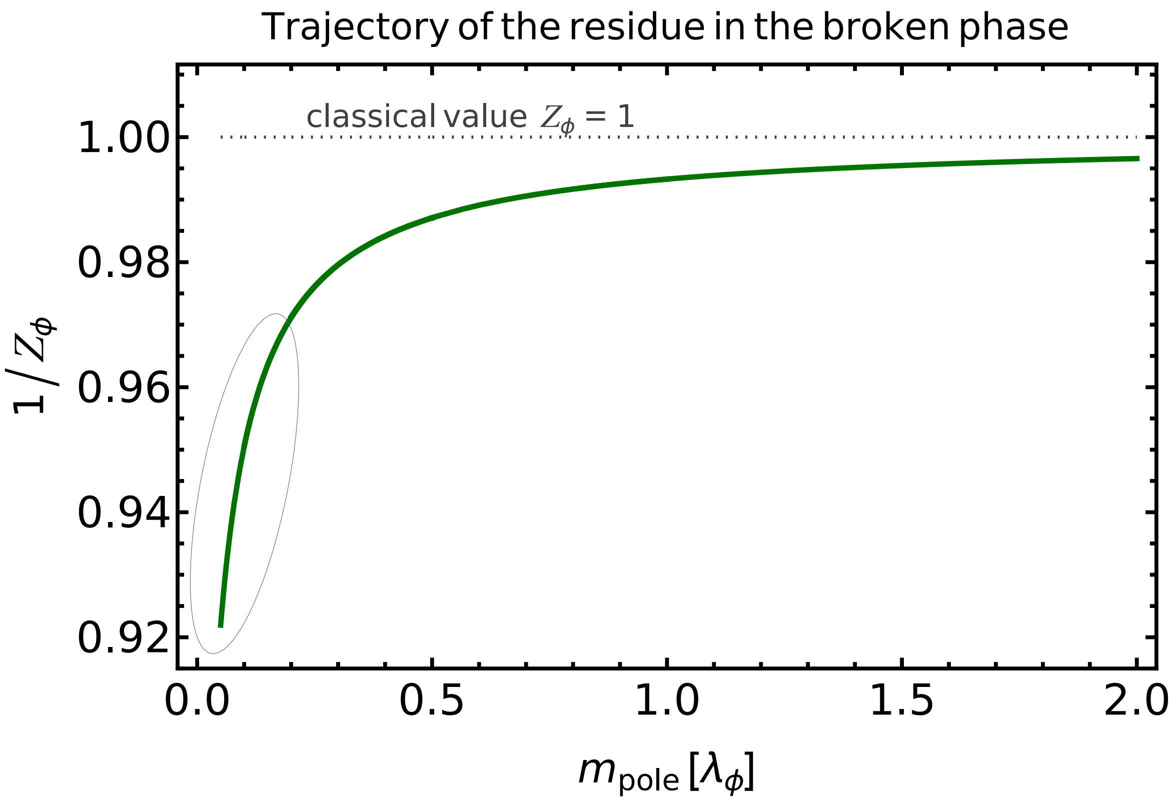
Results in the broken phase

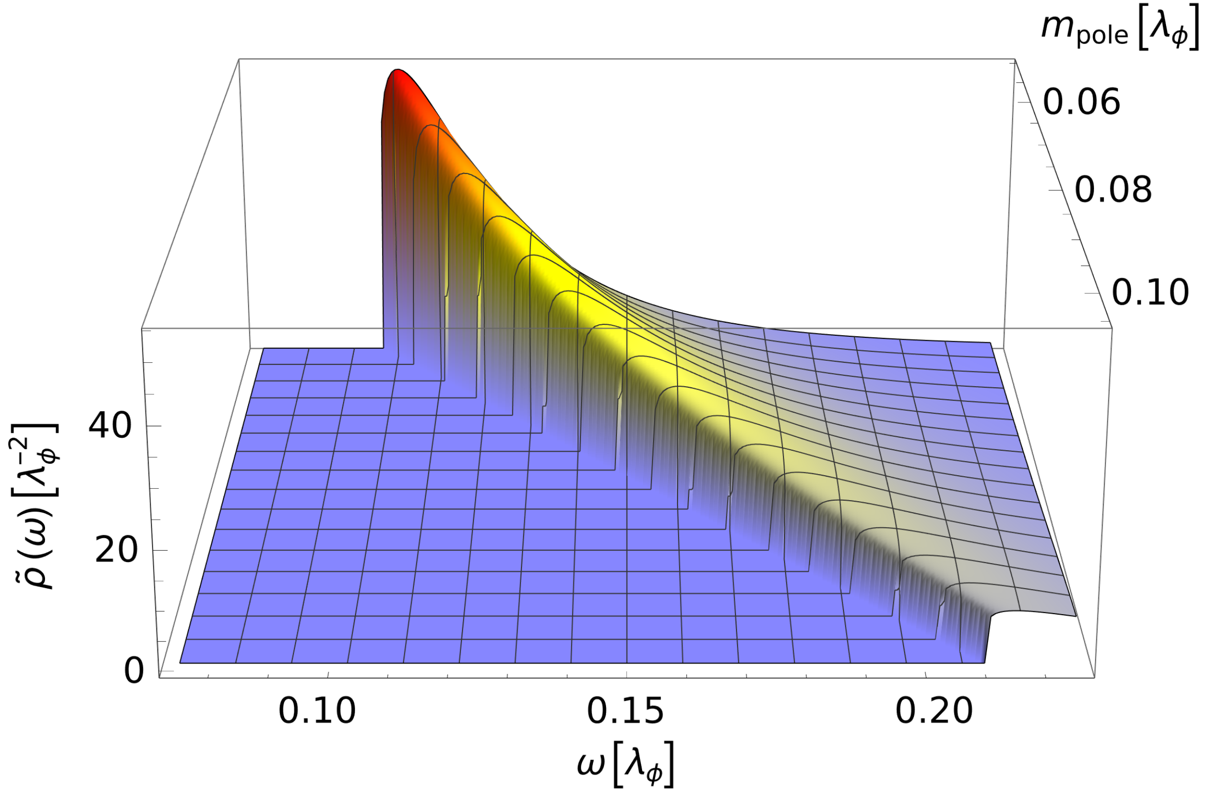




Results in the broken phase

$$\rho(\lambda) = \frac{2\pi}{Z_\phi}\delta(m_\phi^2 - \lambda^2) + \tilde{\rho}(\lambda)$$





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Work to do and open questions – the scaling limit

What happens in the scaling limit?

$$\partial_t \phi_0 = -\frac{\partial_t V_{\text{eff}}^{(1)}(\phi_0)}{V_{\text{eff}}^{(2)}(\phi_0)} = -\frac{\partial_t V_{\text{eff}}^{(1)}(\phi_0)}{m^2_{\text{curv}}} \quad \Rightarrow \quad \phi_0 = \phi_{0,\Lambda}\sqrt{Z_\phi}\left(\frac{k}{\Lambda}\right)\exp\left\{\int_\Lambda^k \frac{dk'}{k'}\mathcal{D}(k')\right\}$$

How to extract critical exponent?

$$\bar{\phi}_0 \propto \tau^\beta\,, \qquad \beta = \frac{1}{2}\nu\,(1+\eta_\phi) \approx 0.3264$$

But what is the tuning parameter?

$$\xi \propto \tau^{-\nu} \qquad \xi \propto k^{-1}$$

$$\tau \propto k^{\frac{1}{\nu}}$$

$$\Rightarrow \quad \bar{\phi}_0 \propto k^{\frac{\beta}{\nu}}\,, \qquad \frac{\beta}{\nu} = \frac{1}{2}(1+\eta_\phi)$$

Work to do and open questions – the scaling limit

How does this RG-prozedure relates to standart RG and scaling analysis?

→ Dynamical mapping between “usual” tuning parameter and on-shell mass

Information on ν in the countertermflow → work in progress

Is this “onshell/comoving” frame a suitable way to think about phase transition?

Wrap-up

- Spectral functional equations are powerful tool to calculate self-consistent spectral functions
- The spectral, functional Callan-Symanzik equation connects correlation functions in the limit of high masses with their massless limit.
- Flowing (on-shell) renomalisation controls trajectory in theory space and can eliminate fine-tuning problems
- Expanding about the (physical) flowing minimum introduces additional diagrams!
- TODO: extend framework to finite temperature and chemical potential
- include the flow of the field-dependent effective potential
- Next goal: self-consistent quark spectral functions at finite T

  → Diffusion coefficients and electric conductivity

Back up

Spectral fRG and the Callan-Symanzik cut-off

arXiv:2206.10232

CS-flow as Limit of finite flows:

$$R^\phi_{k,\Lambda}(p) = Z_\phi\, k^2\, r(x_\Lambda)\,, \qquad x_\Lambda = \frac{\vec{p}^{\,2}}{\Lambda^2} \quad \longrightarrow \quad \lim_{\Lambda\to\infty} R^\phi_{k,\Lambda} = Z_\phi\, k^2$$

Combined RG-transformation with $\Lambda = \Lambda(k)$ and $t_\Lambda = \log(\Lambda/k_{\rm ref})$ and $\mathcal{D}_k = \partial_t \log\Lambda(k)$

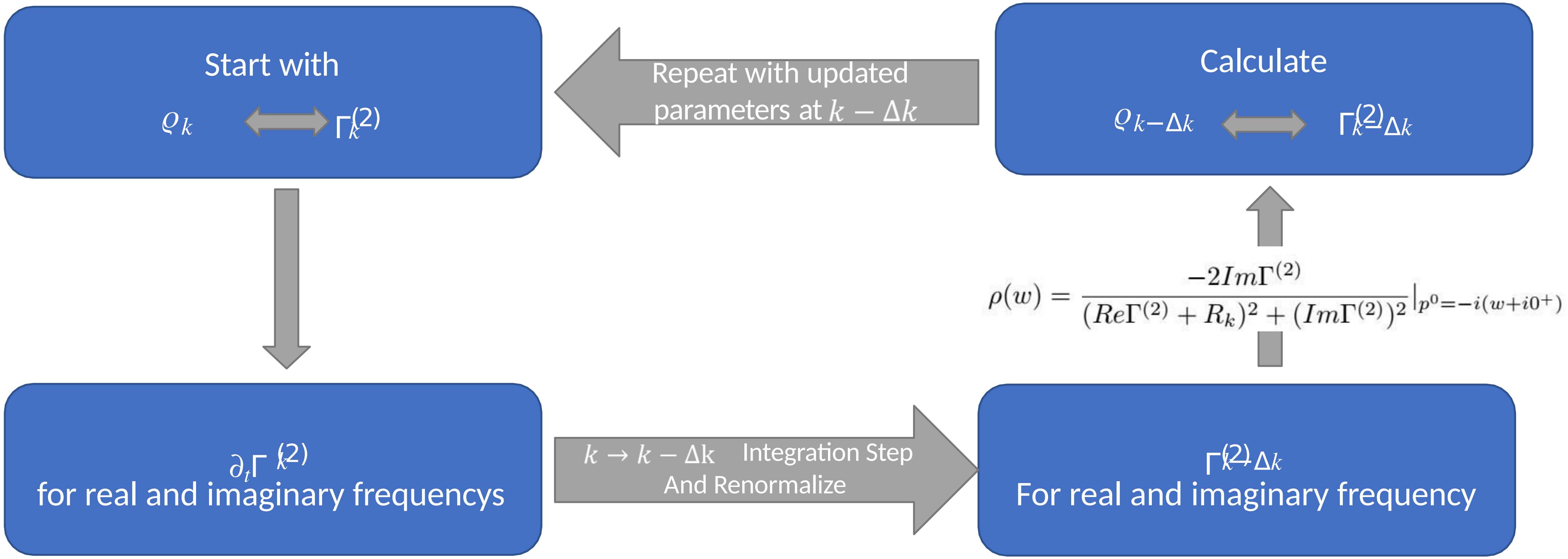
$$\left(\partial_t\big|_\Lambda + \mathcal{D}_k\,\partial_{t_\Lambda}\right)\Gamma_{k,\Lambda} = \frac{1}{2}\mathrm{Tr}\, G^\Phi_{k,\Lambda}\left(\partial_t\big|_\Lambda R^\Phi_{k,\Lambda} + \mathcal{D}_k\,\partial_{t_\Lambda} R^\Phi_{k,\Lambda}\right)$$

$$\partial_t S_{\rm ct}[\phi] := -\frac{1}{2}\mathrm{Tr}\, G^\phi_{k,\Lambda}\,\mathcal{D}_k\,\partial_{t_\Lambda} R^\phi_{k,\Lambda}$$

→ Renormalised CS-equation, counterterms determined by renormalisation condition!

Application to a real scalar field in 3 dimension

local feedback



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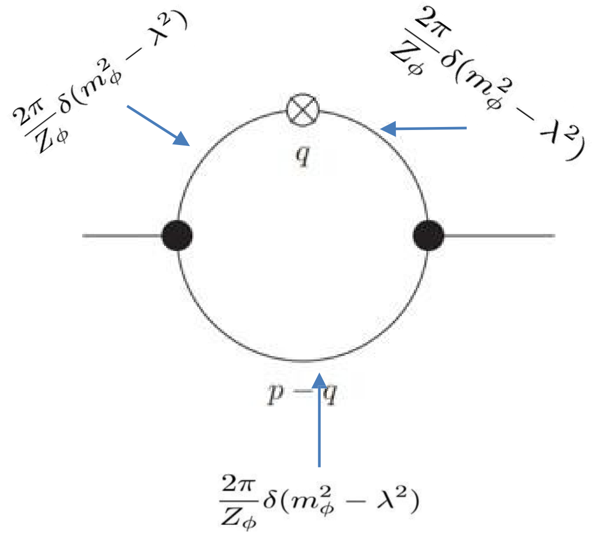
Backup

$$\rho(\lambda) = \frac{2\pi}{Z_\phi}\delta(m_\phi^2 - \lambda^2) + \tilde{\rho}(\lambda)$$

$$I_{\rm pol}(\lambda_1, \lambda_2, \lambda_3, p^2) = \int \frac{d^3q}{(2\pi)^3}\Big(\prod_{i=1}^{2}\frac{1}{(q^2+\lambda_i^2)}\Big)\frac{1}{((q+p)^2+\lambda_3^2}$$



Pole-contributions:

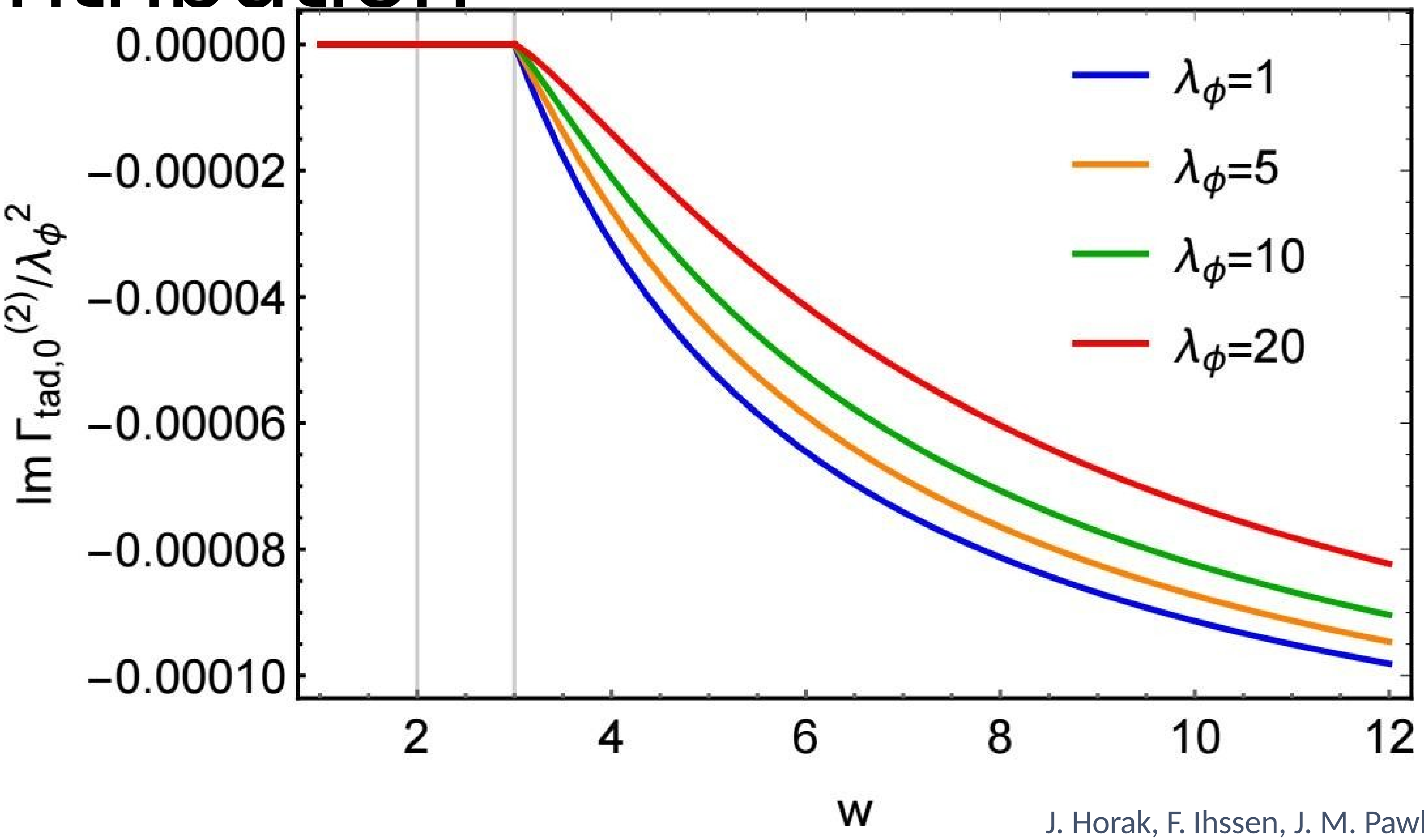
$\lambda_1^2 = \lambda_2^2 = \lambda_3^2 = k^2$

$$\frac{1}{(\lambda^2+q^2)^2} = \frac{-1}{2\lambda}\partial_\lambda \frac{1}{(\lambda^2+q^2)}$$

$$\partial_k \Gamma^{(2)}_{k,1^{st}-order}(p^2) = -\frac{1}{2}\frac{(\Gamma_k^{(3)})^2}{Z_k^3}\frac{1}{4\pi p}\partial_k I_{pol}^{DSE}(m_k, m_k, p^2)$$

$$\int_k^{k_0}\partial_k\Gamma_k^{(2)}(p^2) = -\frac{1}{2}\left[\mathcal{F}(k)I_{\rm pol}^{DSE}(m_k,m_k,p^2)\right]_k^{k_0} + \frac{1}{8\pi p}\int_k^{k_0} dk\,\partial_k\mathcal{F}(k)\,\mathrm{Arctan}[\frac{p}{2m_k}]$$

Results: 1-cut (dynamical) Tadpole contribution



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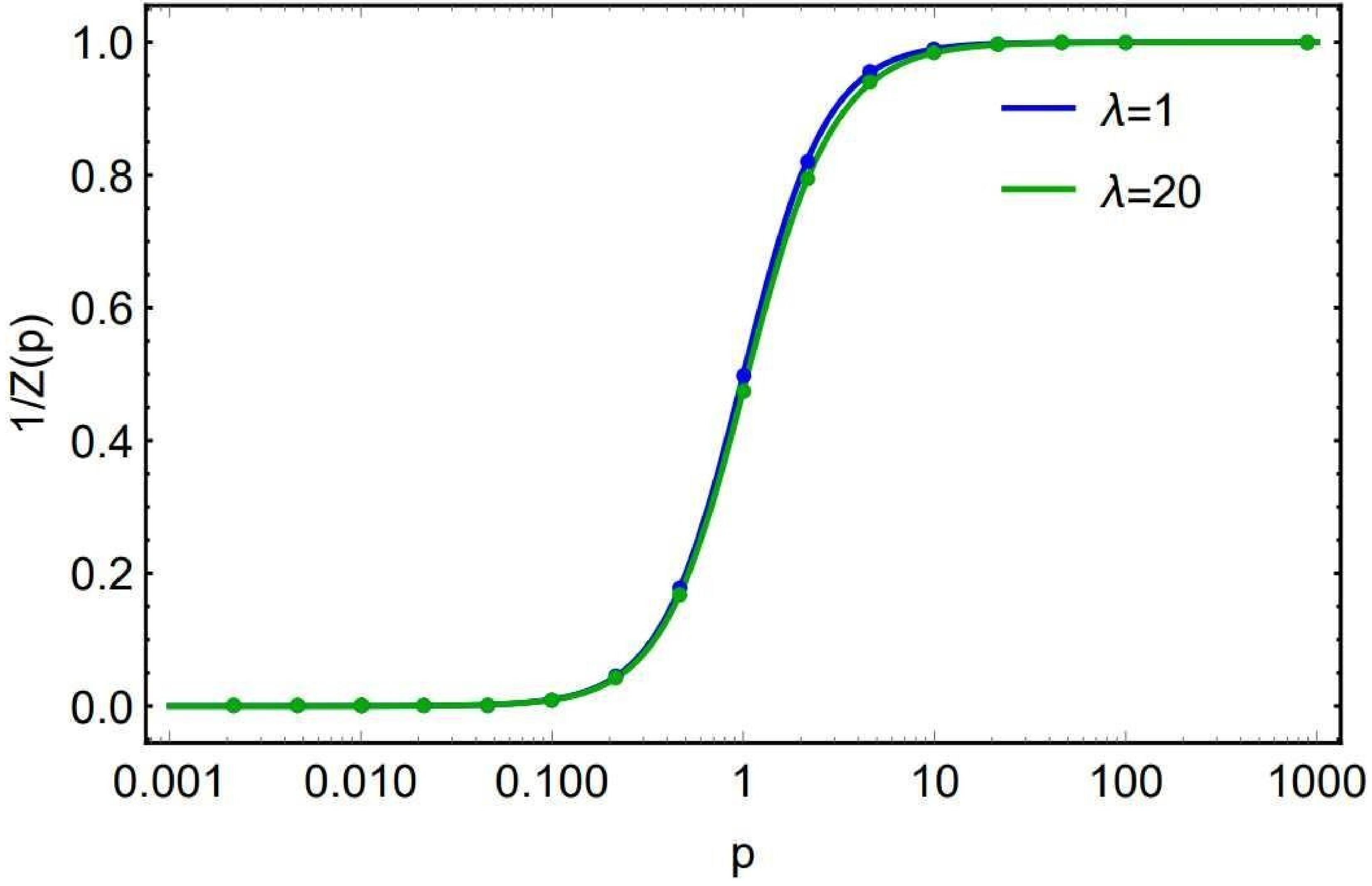
Results: Propagator dressing on the euclidean Axis



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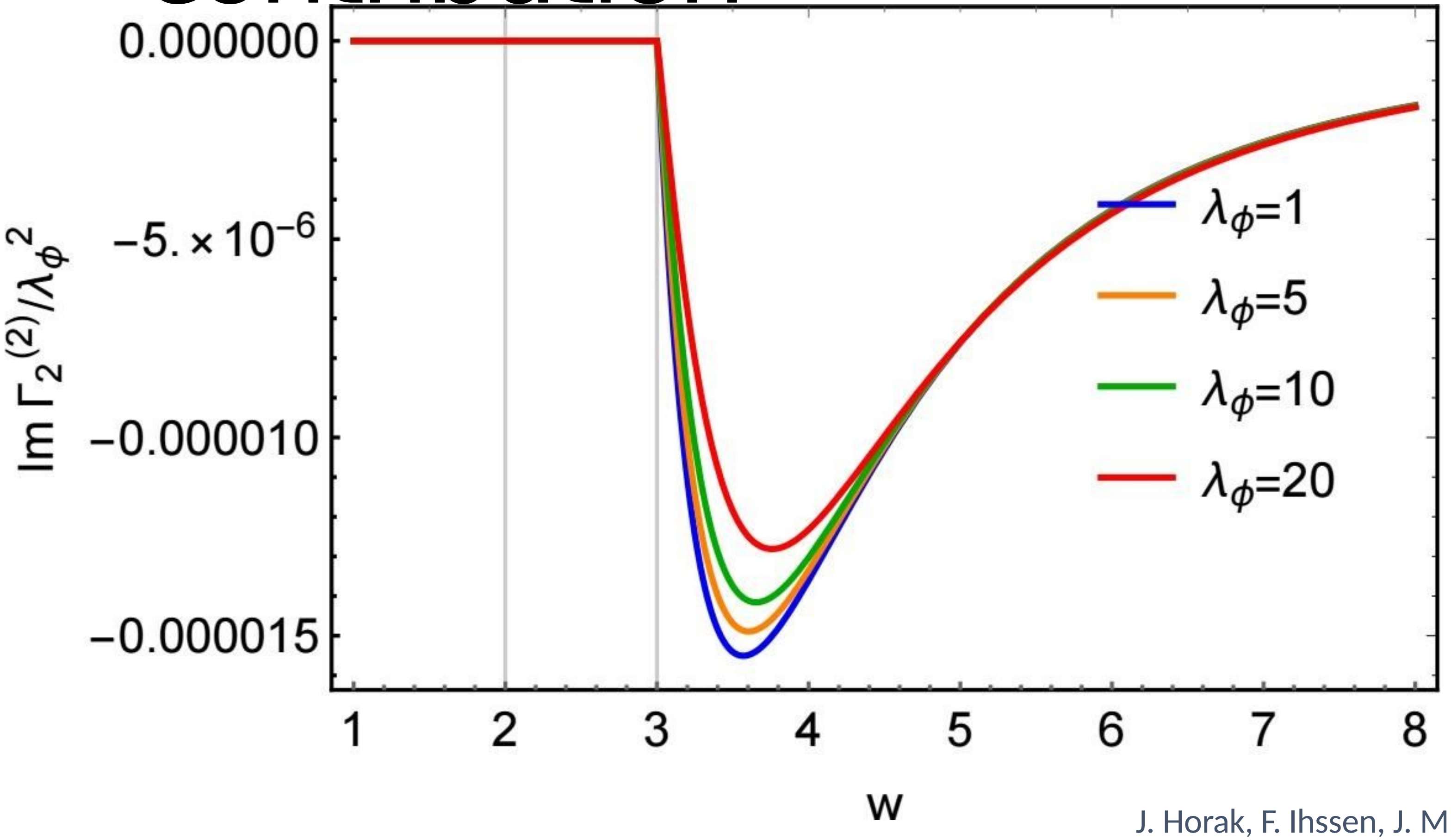
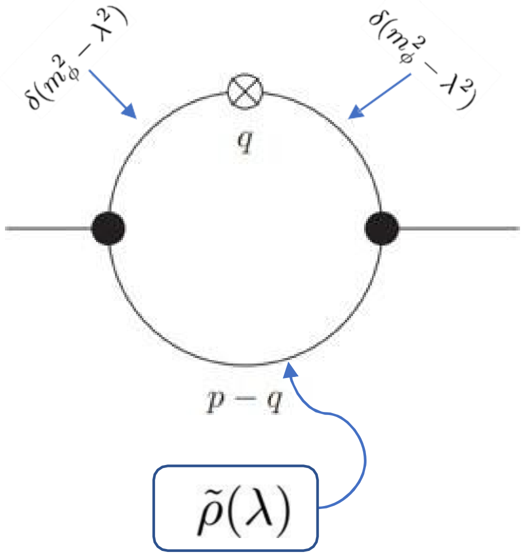
Results: 1-cut contribution





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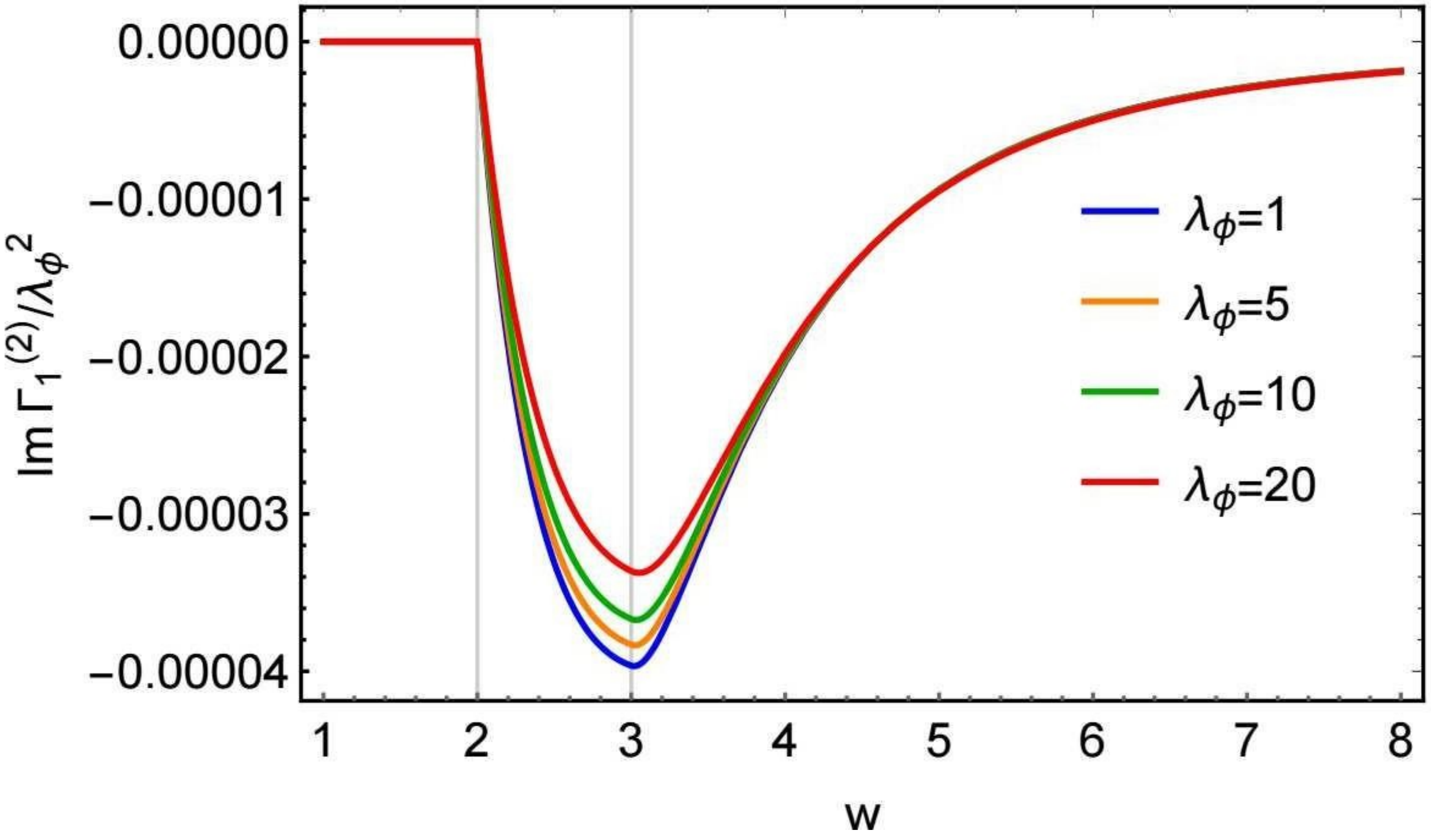
Results: 1-Cut contribution (1)



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