Universality frontier - critical phenomena in the complex plane

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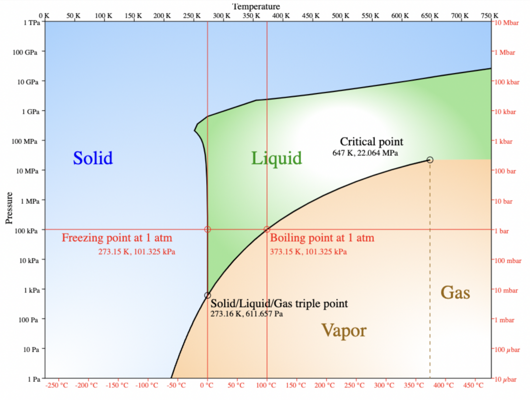
NC STATE UNIVERSITY

The phase diagram of water



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Features:

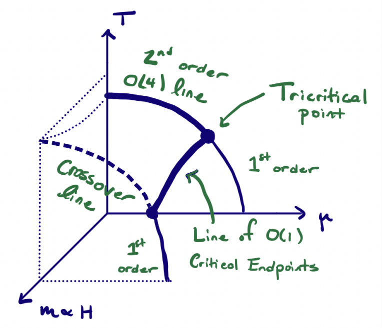
- 1^{st} order and 2^{nd} order phase transitions
- Critical opalescence at the critical point (diverging correlation length)
- Critical point is identical* to that of the Ising model!

  $\rightsquigarrow$ **Universality**

The rich structure of $O(N)$ symmetric field theories

$$S_\Lambda[\phi] = \int d^d x \left\{ \frac{1}{2}(\nabla\phi(\mathbf{x}))^2 + \frac{1}{2} t_\Lambda\, \phi(\mathbf{x})^2 + \frac{1}{4} u_\Lambda\, \phi(\mathbf{x})^4 - H \cdot \phi(\mathbf{x}) \right\} \tag{1}$$

- ϕ is an N-component vector (similarly for H)
- Arise from lattice models near critical points:
- Exhibit Ising-like 2$^{\text{nd}}$ order phase transitions: $\rightsquigarrow$ $O(N)$ Wilson-Fisher (WF) points
- **Universality:** $O(N)$ WF points are prolific!
- Ising $N = 1$, Superfluid Helium $N = 2$, Ferromagnets $N = 3$, chiral* QCD $N = 4$, Random walks $N = 0, -2$



Stephanov, M. A., Phys.Rev.D 73 (2006) 094508

The Landau-Ginzburg approximation and mean-field

$$S_\Lambda[\phi] \;=\; \int_{\mathbf{x}} \left\{ \frac{1}{2} (\nabla\phi(\mathbf{x}))^2 + \frac{1}{2} t_\Lambda \, \phi^2(\mathbf{x}) + \frac{1}{4} u_\Lambda \, \phi^4(\mathbf{x}) - H \cdot \phi(\mathbf{x}) \right\} \qquad (2)$$

$$\mathcal{Z} \;=\; \int_\Lambda \mathcal{D}\phi \, e^{-S_\Lambda[\phi]} \qquad \Lambda \sim a^{-1} \;\; \text{where } a \text{ is a microscopic length scale} \qquad (3)$$

- **Landau Ginzburg Approximation**: Intuition - maintain locality and symmetries

$$\Gamma[\varphi] \;\approx\; \int_{\mathbf{x}} \{ \frac{1}{2} Z \, (\nabla\phi(\mathbf{x}))^2 + \frac{1}{2} t \, \phi(\mathbf{x})^2 + \frac{1}{4} u \, \phi(\mathbf{x})^4 - H \cdot \varphi(\mathbf{x}) \} \qquad (4)$$

- **Mean-Field**: minimizing $\Gamma \rightsquigarrow$ spatially constant field configurations $|\phi(\mathbf{x})| = M$

$$\text{specific free energy} \;\; f(M) \;=\; V^{-1} \, \Gamma_{\mathrm{MF}}[M] = \frac{1}{2} t \, M^2 + \frac{1}{4} u \, M^4 - H \, M \qquad (5)$$

$$\text{extremizing} \;\; f'(M) \;=\; 0 \qquad (6)$$

$$\text{stability} \;\; f''(M) \;\geq\; 0 \;\; \left(\text{Criticality}: \;\; f''(M) = 0\right) \qquad (7)$$

Wilson-Fisher critical points at mean-field

- **The mean-field free energy:**

$$f(M) = \frac{1}{2} t\, M^2 + \frac{1}{4}\, M^4 - H\, M$$

$$f'(M_0) = 0 \;\; \rightsquigarrow \;\; H = M_0\, \left(t + M_0^2\right)$$

- **Reduced temperature:** $t \propto T - T_c$

- **Correlation length:** $\xi = m_{eff}^{-1}$ where $m_{eff}^2 = f''(M_0)$
  $\rightsquigarrow$ 2^{nd} order transition when $f''(M_0) = 0$

- **Criticality:** Solve $f''(M_0) = t + 3M_0^2 = 0$ for t=0 yields H=0

Yang-Lee edge singularity at mean-field

- **The mean-field free energy:**

$$f(M) = \frac{1}{2} t\, M^2 + \frac{1}{4} M^4 - H\, M$$

$$f'(M_0) = 0 \quad \rightsquigarrow \quad H = M_0 \left(t + M_0^2\right)$$

- For $t < 0$: $f''(M_0) = t + 3M_0^2 = 0$ yields H real and non-zero $\sim$ Spinodal point
- **Criticality in the complex plane:** allow for complex fields
- For $t > 0$: $f''(M_c) = t + 3M_c^2 = 0$ yields H purely imaginary $\sim$ Yang-Lee edge singularity (YLE)
- **The Yang-Lee edge location:** H_c and M_c are both purely imaginary

Emergence of a cubic theory

- **The mean-field free energy:**

$$f(M) = \frac{1}{2} t\, M^2 + \frac{1}{4} M^4 - H\, M$$

- **Re-expanded about the Yang-Lee edge:**

$$f(M) = i\, |M_c|\, (M - M_c)^3 - (H - H_c)(M - M_c) + \mathcal{O}(M - M_c)^4$$

- Tuning $h \to h_c$ brings one to the YLE - for $t > 0$
- No longer a $\lambda\, \phi^4$ theory but an $i\, g\, \phi^3$ theory!
- To truly understand the Yang-Lee edge, we need Lee-Yang zeros!

The Lee-Yang theory of phase transitions I

- Consider the Ising model:

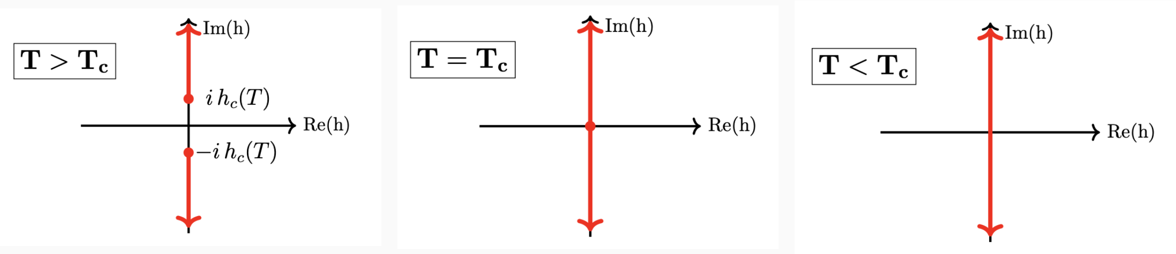
$$\mathcal{H} = -\sum_{\langle i,j\rangle} J s_i s_j - H \sum_i s_i \tag{8}$$

- Factorize the partition function via its zeros:

$$\mathcal{Z}_N(\beta, H) = e^{-\beta N H} \sum_{n=0}^{N} a_n z^n \sim \prod_{i=1}^{N} (z - z_i) \quad \text{where} \quad z = e^{-2\beta H} \tag{9}$$

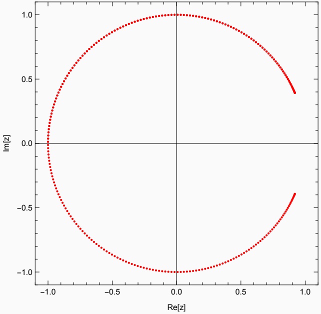
- Thermodynamic limit: $\mathcal{F} \sim \int d\omega\, g(\omega, t) \log(z - \omega)$ where $t \propto T - T_c$
  - non-analyticitites in the density of zeros $g(\omega, t) \rightsquigarrow$ non-analyticities in $\mathcal{F}$
  - Lee-Yang circle theorem: Ising-like* $\rightsquigarrow z_n = e^{i\,\theta_m} \rightsquigarrow$ purely imaginary H

The Lee-Yang theory of phase transitions II

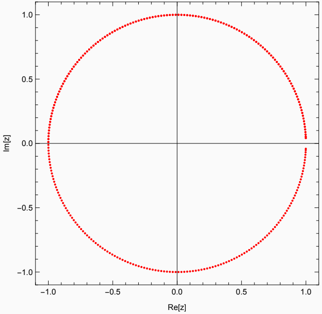


The 1D Ising Model - distribution in the fugacity $z = e^{-2\beta H}$

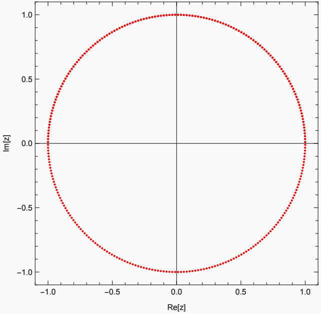
$\mathbf{T > T_c}$:



$\mathbf{T \approx T_c}$:

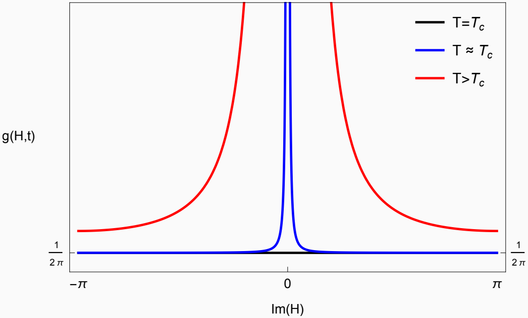


$\mathbf{T = T_c = 0}$:

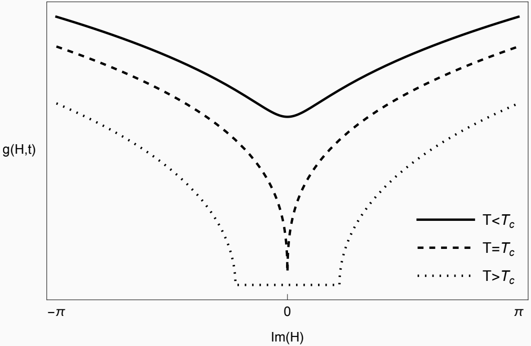


The density of zeros II

The 1D Ising Model:



The mean-field approximation:



- Near the edge: $g(H, t > 0) \propto (H - H_c)^\sigma$

The density of zeros III

Relation between density of zeros $g(H,t)$ and the magnetization:

$$\mathrm{Re}\,[M(H,t)] = 2\pi\, g(H,t) \quad \text{where} \quad H \in i\,\mathbb{R} \tag{10}$$

$$M(0,t) = 2\pi\, g(0,t) \tag{11}$$

- **High temps:** $t > 0$, density vanishes about origin - $g(|H| < |H_c|, t) = 0 \rightsquigarrow M \in i\,\mathbb{R}$
- **Low temps:** $t < 0$, density is finite about origin which produces $M(H = 0, t) \neq 0$!
- **Edge scaling:** $g(H, t > 0) \propto (H - H_c)^{\sigma} \rightsquigarrow M - M_c \propto (H - H_c)^{\sigma}$

The Yang-Lee edge singularity I

$$f(M) = i\,|M_c|\,(M - M_c)^3 - (H - H_c)(M - M_c)$$

$$S = \int_{\mathbf{x}} \left\{ \frac{1}{2}\,(\nabla\varphi)^2 + \frac{1}{3} ig\,\varphi^3 - (H - H_c)\varphi \right\}$$

- Critical Data: $\varphi \sim M - M_c$

$$\varphi \propto (H - H_c)^{\sigma} \quad (12)$$

$$\sigma = \frac{d - 2 + \eta}{d + 2 - \eta} \quad (13)$$

- Imaginary coupling $\rightsquigarrow \eta < 0$.
- $H \neq 0 \rightsquigarrow \sigma$ is identical $\forall\ O(N)$ models.
- $\phi^3 \rightsquigarrow$ upper critical dimension $d_c = 6$.

The thinking: non-analyticities in $g(H,t)$ yield critical points

The Wilson-Fisher Point (WF):

- Occurs at $t = H = 0$

$$S = \int_{\mathbf{x}} \left\{ \frac{1}{2} (\nabla\phi)^2 + \frac{1}{2} r\,\phi^2 + \frac{1}{4} u\,\phi^4 - H\phi \right\}$$

The Yang-Lee Edge Singularity (YLE):

- Occurs at $t > 0$ and $H = \pm H_c \in i\,\mathbb{R}$

$$S = \int_{\mathbf{x}} \left\{ \frac{1}{2} (\nabla\varphi)^2 + \frac{1}{3} ig\,\varphi^3 - (H - H_c)\varphi \right\}$$

- **Idea A:** For $t \ll 1$, YLE $\in$ critical region of WF $\rightsquigarrow$ extended universality.
- **Idea B:** The dependence of the gap H_c on $T - T_c$ is universal (in appropriate variables).

Scaling and universality near critical points I

- **Big Idea:** Critical points (static) belong to **universality classes** labelled by the underlying symmetry and the dimension.

  Example: QCD's critical endpoint*, the liquid-gas critical point, and the Ising model critical point $\rightsquigarrow$ Ising Universality Class.

- The proper framework for scaling is via the Renormalization Group (RG):

  $\rightsquigarrow$ homogeneous free energy and correlation functions in t and h (scaling invariance):

$$f_s(t,h) = b^{-d} f_s(t\, b^{y_t},\, h\, b^{y_h}) \tag{14}$$

- Take $h\, b^{y_h} = 1 \rightsquigarrow b = h^{-1/y_h}$: universal scaling function $F_s(z)$

$$\begin{aligned} f_s(t,h) &= h^{d/y_h} f_s(t\, h^{-y_t/y_h}, 1) & (15) \\ &= h^{d/y_h} F_s(z = t\, h^{-y_t/y_h}) & (16) \end{aligned}$$

Scaling and universality near critical points II

- Free energy of any member of a universality class $\rightsquigarrow$ a universal scaling function $F_s(z)$:

$$f_s(t,h) \quad = \quad h^{d/y_h}\, F_s(z = t\, h^{-y_t/y_h}) \tag{17}$$

- Universal magnetic equation of state:

$$M = \frac{\partial f_s}{\partial h} \quad \rightsquigarrow \quad M = h^{1/\delta} f_G(z) \quad \text{where} \quad z = t\, h^{-1/\beta\delta} \tag{18}$$

RG Eigenvalues → Critical Exponents:

- $\delta = \dfrac{y_h}{d - y_h}, \dfrac{1}{\beta\delta} = \dfrac{y_t}{y_h}$
- Critical amplitudes are needed to define the rescaled variables t, h.

More on Universality:

- All thermodynamic quantities have scaling functions.
- **Powerful** to know details of these functions.
- Most* details have been heavily studied.

Extended universality from the Yang-Lee edge: Two views

Rescale t and H:

$$M = B\,(-t)^{\beta} \rightsquigarrow M = \left(-\bar{t}\,\right)^{\beta}$$
$$M = B_c\,H^{1/\delta} \rightsquigarrow M = h^{1/\delta}$$

Universal magnetic equation of state:

$$M = h^{1/\delta}\,f_G\,(z)$$
$$z = \bar{t}\,h^{-1/\beta\delta}$$

Universal Amplitude* Ratio:

$$H_c = i\,H_0\,t^{\,\beta\delta} \rightsquigarrow h_c = i\,R_h\,\bar{t}^{\,\beta\delta}$$

$$R_h = \frac{B_c^{\delta}\,H_0}{B^{\delta}}$$

Universal Location:

$$z_c = t\,h_c^{-1/\beta\delta}$$

$$\mathrm{Arg}(z_c) = \frac{\pi}{2\beta\delta} \text{ but } |z_c| =?$$

Important? Yes! It determines analytic structure, radius of convergence, etc.

Another useful quantity: $\zeta_c = \dfrac{z_c}{R_{\chi}^{1/\gamma}}$

A moment for pause

- YLE critical exponent σ (and η) - well studied: in d=3, $\sigma \approx 0.085$
- YLE universal location - previously unknown for the physically relevant cases of $d = 3$!

- 1d Ising: $(z_c, \sigma) = (1, -\frac{1}{2})$
- 2d Ising: $(z_c, \sigma) \approx (3.9554, -\frac{1}{6})$
- Mean-field ($d > 4$): $(z_c, \sigma) = (\frac{3}{2^{2/3}}, \frac{1}{2})$
- Large N ($2 < d < 4$):

Fonseca, P. and Zamolodchikov, A., arXiv:hep-th/0112167

- **Impact of determining z_c for 3d $O(N)$ models:**
  - First calculation of a non-trivial universal quantity z_c.
    Connelly,A., GJ, Rennecke, F, Skokov, V., PhysRevLett.125.191602
  - $O(4)$ $z_c \rightsquigarrow$ radius of convergence of $\mu = 0$ Taylor expansion for lattice QCD - constraints on QCD's critical endpoint.
    Connelly, A; GJ, Mukherjee, S; Skokov, S, Nucl.Phys.A 1005 (2021) 121834
  - Precision results for important 3d $O(N)$ models: $N = 1 - 5$
    GJ, Rennecke, F., Skokov, V., arXiv:2211.00710

The functional renormalization group (FRG)

- At the microscopic scale Λ: $\Gamma_{\mathrm{LG}}[\varphi] \approx S_\Lambda[\varphi] \sim$ no fluctuations (classical action!).
- Question: How do fluctuations on length scales larger than Λ^{-1} alter $\Gamma_{\mathrm{LG}}[\varphi]$?
- FRG: Construct a scale-dependent action $\Gamma_k[\varphi]$ by including a regulator R_k - a scale dependent mass term - which suppresses the fluctuations with $p < k$ to $\Gamma_k[\varphi]$.
- Requirements of the Regulator R_k:

  1.) $\Gamma_{k=\Lambda}[\varphi] = S_\Lambda[\varphi]$, 2.) $\Gamma_{k=0}[\varphi] = \Gamma[\varphi] \sim$ the full effective action.
- The Functional (Exact) Renormalization Group Equation: $t = \ln(k/\Lambda)$

$$\partial_t \Gamma_k[\varphi] = \frac{1}{2} \mathrm{Tr} \left\{ \partial_t R_k \left(\frac{\delta^2 \Gamma_k[\varphi]}{\delta\varphi_i\, \delta\varphi_j} + R_k \right)^{-1} \right\}.$$

Quick Sketch: Leading order of the derivative expansion

- The local potential approximation (LPA): $\rho = \dfrac{1}{2}\phi_i\phi^i$

$$\Gamma_k[\varphi] \quad = \quad \int d^d x \left\{ \frac{1}{2}(\nabla\varphi)^2 + U_k(\rho) \right\}. \tag{19}$$

- Using the regulator $R_k = (k^2 - p^2)\Theta(k^2 - p^2)$:

$$\partial_t U_k(\rho) \quad = \quad \frac{2^{-(d-1)}\, k^{d+2}}{d\,\pi^{d/2}\,\Gamma\left(\frac{d}{2}\right)} \left(\frac{1}{k^2 + U_k' + 2\rho\, U_k''} + \frac{N-1}{k^2 + U_k'} \right). \tag{20}$$

FRG within the derivative expansion II

Next-to-leading order in the derivative expansion: $\rho = \frac{1}{2}\phi_i\phi^i$

$$\Gamma_k[\phi] = \int_x \left\{ U_k(\rho) + \frac{1}{2}Z_k(\rho)\left(\partial\phi\right)^2 + \frac{1}{4}Y_k(\rho)\left(\partial\rho\right)^2 \right\}$$

- Taylor expand about fixed renormalized mass: $Z_k^{-1}\,U_k''(\phi_{0,k}) = m_R^2 = \text{const}$
- Renormalize via Radial Field Renorm. $Z_k = Z_k(\phi_{0,k})$ and $Y_k = Y_k(\phi_{0,k})$
  - $Z_{\perp,k} = Z_k$
  - $Z_{\parallel,k} = Z_k + \rho\,Y_k \rightsquigarrow$ Basis: $U_k^{(n)}$, $Y_k^{(n)}$, $Z_{\parallel,k}^{(n)}$
  - Define the running anomalous dimension: $\eta_k = -\partial_t \ln Z_{\parallel,k}$
- Numerically solving flows: choose $R_k = \alpha(k^2 - p^2)\Theta(k^2 - p^2)$

Scaling Solutions vs Explicit Solving of Flow Equations

- For critical physics, it is best to use dimensionless renormalized variables:
  - $\bar{\rho} = k^{d-2}\, Z_k^{-1}\, \rho$
  - $\bar{U}_k^{(n)}(\bar{\rho}) = k^{d-n(d-2)}\, Z_k^n\, U_k^{(n)}(\rho)$

Scaling solutions:

- Scale invariance (fixed point)

  $\rightsquigarrow \partial_t \bar{U}_*(\bar{\rho}) = 0$ (algebraic eqns.)
- Similar equations for $\bar{Z}_k(\bar{\rho})$, $\bar{Y}_k(\bar{\rho})$, ...
- Perturbations about $\bar{U}_*(\bar{\rho}) \rightsquigarrow$ critical exponents (linearized beta functions)
- $\bar{U}_*(\bar{\rho})$ does not give critical amplitudes!

  Need physical Infrared (IR) potential $U_{k=0}(\rho)$ for the amplitudes

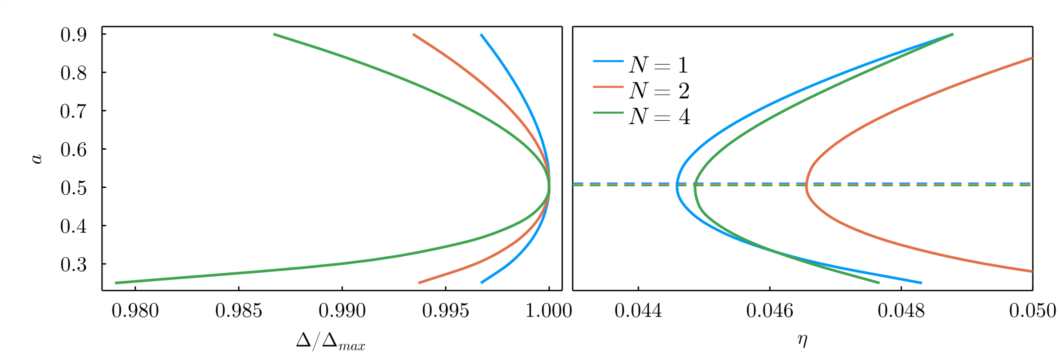
Explicit flow solutions:

- Perturb about fixed point solution
- Numerically solve diff. eqns. for $k \in (0, \Lambda]$
- IR physics $\rightsquigarrow$ exponents AND amplitudes
- Use m_R^2 to probe appropriate regions of complex H-plane:

  $H > 0$ vs. $H = 0$ vs. $H \in i\,\mathbb{R}$

Process for computing ζ_c at $d = 3$ for $O(N)$ models with $N = 1 - 5$

Step 1: Principle of minimal sensitivity (PMS): Apply to $\Delta = \beta\delta$ via WF scaling solutions.



FRG at $\mathcal{O}(\partial^2)$: the process II

Step 1: $\Delta = \beta\delta$ PMS location $\rightsquigarrow$ fix α for the IR flows.

Step 2: Thermal and massive perturbations to WF.

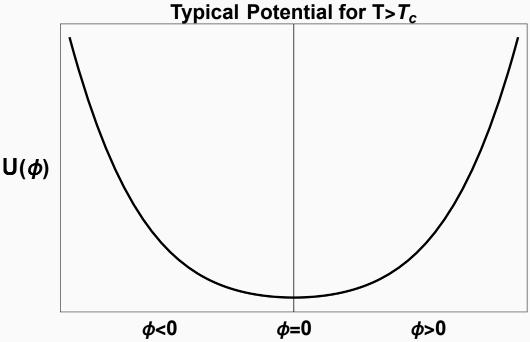
Question: How to tune the mass to see WF and YLE?

High Temp. $t > 0$:

$U''_{k=0}(\phi = 0) = m^2_{0,+} > 0$

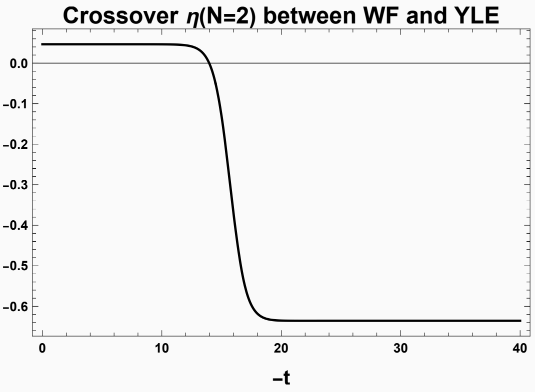
$m^2_{0,+} < m^2_R \quad \rightsquigarrow M_{k=0} \in \mathbb{R}$

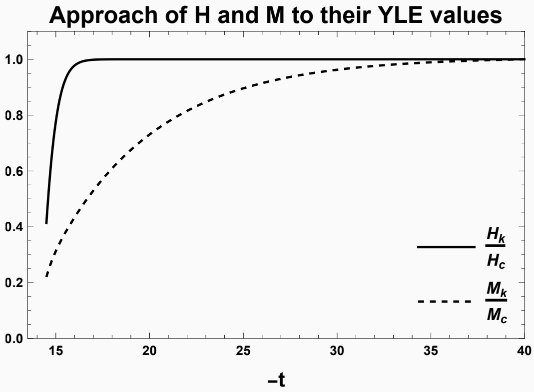
$0 < m^2_R < m^2_{0,+} \quad \rightsquigarrow M_{k=0} \in i\,\mathbb{R}!$



Magnetization: $M_k = \phi_{0,k}$

"Running" Magnetic Field: $H_k \equiv U_k'(\phi_{0,k})$





The YLE scaling solution:

$$\partial_t \bar{\varphi}_{0,k} = 0 \rightsquigarrow \partial_t \left(\bar{\phi}_{0,k} - \bar{M}_c\right) = 0$$
$$\partial_t \, \delta \bar{H}_k = 0 \rightsquigarrow \partial_t \left(\bar{H}_k - \bar{H}_c\right) = 0$$

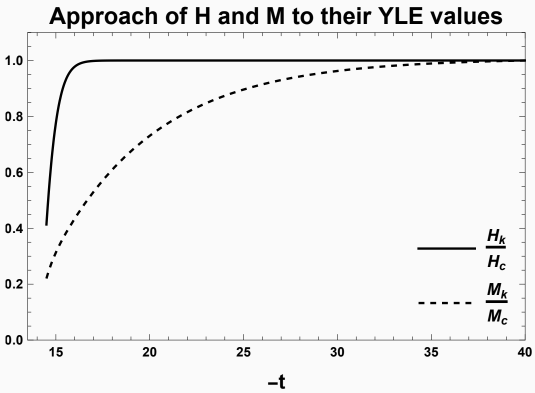
Implication for physical variables:

$$\varphi_{0,k} \sim k^{\frac{d+2-\eta}{2}} \rightsquigarrow \phi_{0,k} - M_c \sim k^{\frac{d-2+\eta}{2}}$$
$$\delta H_k \sim k^{\frac{d+2-\eta}{2}} \rightsquigarrow H_k - H_c \sim k^{\frac{d+2-\eta}{2}}$$

In 3D: $\eta = \eta_{\rm YLE} \sim -.6$

$$\phi_{0,k} - M_c \sim k^{0.2}$$
$$H_k - H_c \sim k^{2.8}$$

$$S = \int_{\mathbf{x}} \left\{ \frac{1}{2} (\partial \varphi)^2 + \frac{1}{3} i g \, \varphi^3 - \delta H \varphi \right\}$$



FRG at $\mathcal{O}(\partial^2)$: the process V

Step 2 cont. Extract B_c and δ; C_2^+ and γ; and H_c

$$M = B_c\, H^{1/\delta} \quad \text{where} \quad t = 0 \tag{21}$$

$$\chi = C_2^+\, t^{-\gamma} \quad \text{where} \quad H = 0 \text{ and } \quad t > 0 \tag{22}$$

a.) Set $t = 0$ and $m_R^2 \ll 1 \rightsquigarrow B_c$ and δ

b.) For a range of $t > 0$, find $m_{0,+}^2(t) \rightsquigarrow C_2^+$ and γ

c.) Compute H_c for some sufficiently small t

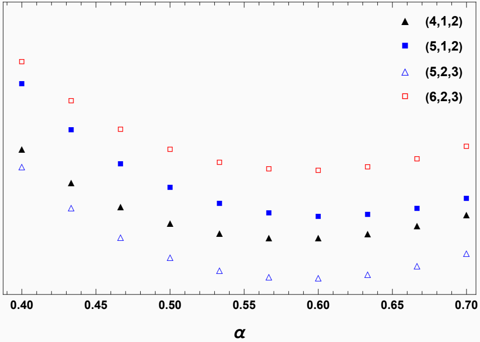
Done*: $\zeta_c = \dfrac{z_c}{R_\chi^{1/\gamma}} = \left(\dfrac{B_c}{C_2^+}\right)^{1/\gamma} t\, H_c^{-1/\beta\delta}$

Step 3: Systematic error measure: Additional PMS

Repeat ζ_c computation at H_c PMS

- **Truncations:** $\sim (N_U, N_Y, N_{Z_\parallel})$ in terms of a ρ expansion

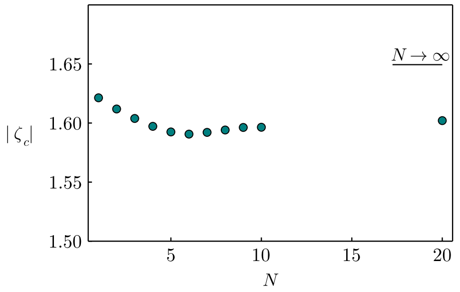
Typical H_c PMS for $N = 2$:



Comparison of PMS locations at (6,2,3):

N	1	2	3	4
α_Δ	0.5108	0.5069	0.5026	0.5044
α_η	0.5044	0.5075	0.5064	0.4906
α_{H_c}	0.6299	0.5921	0.5724	0.5617

The universal location:



- **Truncations:** $\sim (N_U, N_Y, N_{Z_\parallel})$

  $(5,1,2),\ (5,2,3),$ and $(6,2,3)$

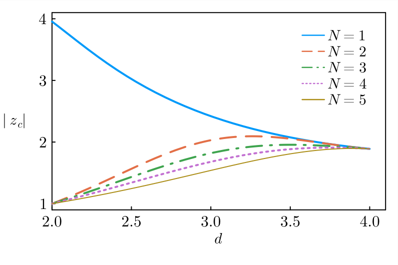
- **Error Analysis:** $(\Delta_{\rm tr}),\ (\Delta_{\rm reg})$
- **Mapping to z_c:** Literature $\rightsquigarrow R_\chi$

N	1	2	3	4	5
$\lvert\zeta_c\rvert$	1.621(4)(1)	1.612(9)(0)	1.604(7)(0)	1.597(3)(0)	1.5925(2)(1)
$\lvert z_c\rvert$	2.43(4)	2.04(8)	1.83(6)	1.69(3)	1.55(4)

Summary and Future Work

Summary:

- Extended universality of the YLE
- A crossover via FRG in the complex plane
- Next-to-leading order result for ζ_c of $O(N)$ universality classes

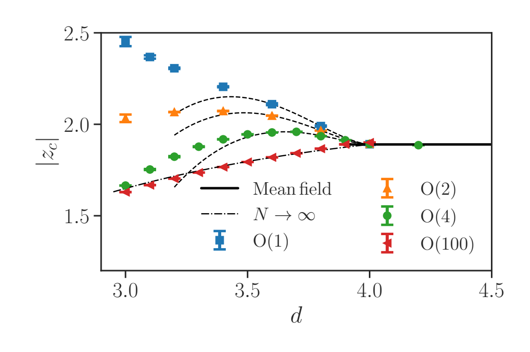


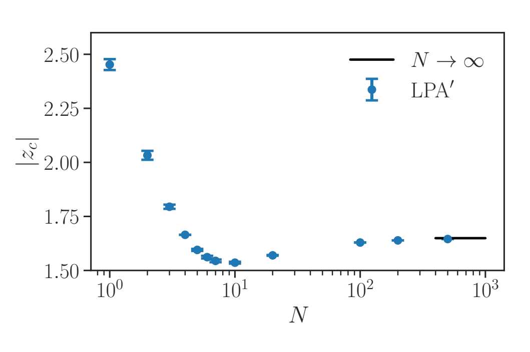
Future Work:

- Compute ζ_c and/or z_c for the $O(N)$ universality classes with $-2 < N < 1$.
- Extract remaining universality of YLE.
- Generalize study to multicritical fixed points and fixed points with fermionic degrees of freedom.

Extrapolation results: universal Yang-Lee edge location at LPA'

Results below were obtained using finite temperature flows:





- Demonstrating that mean-field results become exact in $d = 4$ dimensions, independent of N.
- Demonstrating convergence to $d = 3$ large N limit.

Connelly, A., GJ, Rennecke, F, Skokov, V., PhysRevLett.125.191602