Generalising local Quantum Field Theory to finite temperatures

(Based on: P. Lowdon, R.-A. Tripolt, J. M. Pawlowski, D. H. Rischke, **2104.13413)**

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Outline

1. Local QFT in the vacuum
2. Extension to finite T
3. Spectral implications
4. Shear viscosity in Φ^4 theory
5. Summary & outlook

1. Local QFT in the vacuum

- In the 1960s, A. Wightman and R. Haag pioneered an approach which set out to answer the fundamental question "*what is a QFT?*"
- The resulting approach, "Local QFT", defines a QFT using a core set of physically motivated axioms

Axiom 1 (Hilbert space structure). *The states of the theory are rays in a Hilbert space $\mathcal{H}$ which possesses a continuous unitary representation $U(a,\alpha)$ of the Poincaré spinor group $\overline{\mathscr{P}^{\uparrow}_{+}}$.*

Axiom 2 (Spectral condition). *The spectrum of the energy-momentum operator P^{μ} is confined to the closed forward light cone $\overline{V}^{+} = \{p^{\mu} \mid p^2 \geq 0,\, p^0 \geq 0\}$, where $U(a,1) = e^{iP^{\mu}a_{\mu}}$.*

Axiom 3 (Uniqueness of the vacuum). *There exists a unit state vector $|0\rangle$ (the vacuum state) which is a unique translationally invariant state in $\mathcal{H}$.*

Axiom 4 (Field operators). *The theory consists of fields $\varphi^{(\kappa)}(x)$ (of type (κ)) which have components $\varphi^{(\kappa)}_{l}(x)$ that are operator-valued tempered distributions in $\mathcal{H}$, and the vacuum state $|0\rangle$ is a cyclic vector for the fields.*

Axiom 5 (Relativistic covariance). *The fields $\varphi^{(\kappa)}_{l}(x)$ transform covariantly under the action of $\overline{\mathscr{P}^{\uparrow}_{+}}$:*

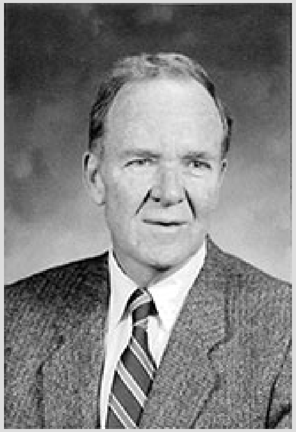
$$U(a,\alpha)\varphi^{(\kappa)}_{i}(x)U(a,\alpha)^{-1} = S^{(\kappa)}_{ij}(\alpha^{-1})\varphi^{(\kappa)}_{j}(\Lambda(\alpha)x + a)$$

where $S(\alpha)$ is a finite dimensional matrix representation of the Lorentz spinor group $\overline{\mathscr{L}^{\uparrow}_{+}}$, and $\Lambda(\alpha)$ is the Lorentz transformation corresponding to $\alpha \in \overline{\mathscr{L}^{\uparrow}_{+}}$.

Axiom 6 (Local (anti-)commutativity). *If the support of the test functions f, g of the fields $\varphi^{(\kappa)}_{l}, \varphi^{(\kappa')}_{m}$ are space-like separated, then:*

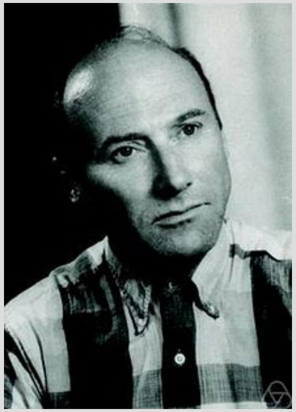
$$[\varphi^{(\kappa)}_{l}(f), \varphi^{(\kappa')}_{m}(g)]_{\pm} = \varphi^{(\kappa)}_{l}(f)\varphi^{(\kappa')}_{m}(g) \pm \varphi^{(\kappa')}_{m}(g)\varphi^{(\kappa)}_{l}(f) = 0$$

when applied to any state in $\mathcal{H}$, for any fields $\varphi^{(\kappa)}_{l}, \varphi^{(\kappa')}_{m}$.



A. Wightman

[R. F. Streater and A. S. Wightman, *PCT, Spin and Statistics, and all that* (1964).]

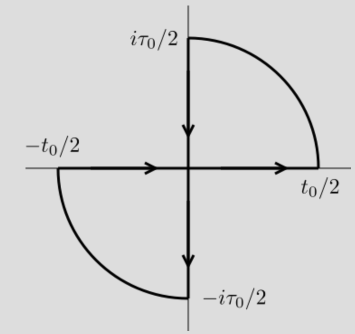


R. Haag

[R. Haag, *Local Quantum Physics*, Springer-Verlag (1996).]

1. Local QFT in the vacuum

- Local QFT has led to many important structural insights, including:

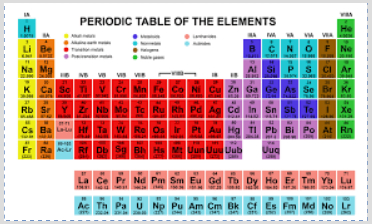


→ The connection of Minkowski and Euclidean QFTs

→ *CPT* is a symmetry of any QFT

→ Spin-statistics theorem

→ Scattering theory



→ Existence of dispersion relations

But... this framework describes QFT in the vacuum state, what about $T > 0$?

- Important progress was made by Bros and Buchholz [Z. Phys. C **55** (1992) 509]

... which was was later built upon [hep-th/9606046, hep-th/9807099, hep-ph/0109136]

2. Extension to finite T

- **Idea**: Look for a generalisation of the standard axioms that is compatible with $T > 0$, and approaches the vacuum case for $T \to 0$

Axiom 1 (Hilbert space structure). *The states of the theory are rays in a Hilbert space $\mathcal{H}$ which possesses a continuous unitary representation $U(a, \alpha)$ of the Poincaré spinor group $\overline{\mathscr{P}_+^\uparrow}$.*

→ **H_β is defined for fixed $\beta = 1/T$**

Axiom 2 (Spectral condition). *The spectrum of the energy-momentum operator P^μ is confined to the closed forward light cone $\overline{V}^+ = \{p^\mu \mid p^2 \geq 0,\, p^0 \geq 0\}$, where $U(a, 1) = e^{iP^\mu a_\mu}$.*

→ **Replaced by the KMS condition**

$$\langle\Omega_\beta|\phi(x_1)\cdots\phi(x_k)\phi(x_{k+1})\cdots\phi(x_n)|\Omega_\beta\rangle$$
$$= \langle\Omega_\beta|\phi(x_{k+1})\cdots\phi(x_n)\phi(x_1 + i(\beta,\vec{0}))\cdots\phi(x_k + i(\beta,\vec{0}))|\Omega_\beta\rangle$$

Axiom 3 (Uniqueness of the vacuum). *There exists a unit state vector $|0\rangle$ (the vacuum state) which is a unique translationally invariant state in $\mathcal{H}$.*

→ **Instead, thermal background state $|\Omega_\beta>$**

Axiom 4 (Field operators). *The theory consists of fields $\varphi^{(\kappa)}(x)$ (of type (κ)) which have components $\varphi_l^{(\kappa)}(x)$ that are operator-valued tempered distributions in $\mathcal{H}$, and the vacuum state $|0\rangle$ is a cyclic vector for the fields.*

✓ **Fields are still distributions**

Axiom 5 (Relativistic covariance). *The fields $\varphi_l^{(\kappa)}(x)$ transform covariantly under the action of $\overline{\mathscr{P}_+^\uparrow}$:*

$$U(a,\alpha)\varphi_i^{(\kappa)}(x)U(a,\alpha)^{-1} = S_{ij}^{(\kappa)}(\alpha^{-1})\varphi_j^{(\kappa)}(\Lambda(\alpha)x + a)$$

where $S(\alpha)$ is a finite dimensional matrix representation of the Lorentz spinor group $\overline{\mathscr{L}_+^\uparrow}$, and $\Lambda(\alpha)$ is the Lorentz transformation corresponding to $\alpha \in \overline{\mathscr{L}_+^\uparrow}$.

→ **The fields no longer transform under general unitary Lorentz transformations**

Axiom 6 (Local (anti-)commutativity). *If the support of the test functions f, g of the fields $\varphi_l^{(\kappa)}, \varphi_m^{(\kappa')}$ are space-like separated, then:*

$$[\varphi_l^{(\kappa)}(f), \varphi_m^{(\kappa')}(g)]_\pm = \varphi_l^{(\kappa)}(f)\varphi_m^{(\kappa')}(g) \pm \varphi_m^{(\kappa')}(g)\varphi_l^{(\kappa)}(f) = 0$$

when applied to any state in $\mathcal{H}$, for any fields $\varphi_l^{(\kappa)}, \varphi_m^{(\kappa')}$.

✓ **Locality is unaffected by the properties of the background state. *This is important!***

3. Spectral implications

- For simplicity, consider a thermal QFT involving real scalar fields $\Phi(x)$

- It turns out that by demanding the fields be ***local***, this imposes significant constraints on the structure of the correlation functions!

  $\rightarrow$ In particular, the thermal commutator has the general form:

$$\widetilde{C}_{\beta}(p_0,\vec{p}) := \mathcal{F}\left[\langle\Omega_{\beta}|\,[\phi(x),\phi(y)]\,|\Omega_{\beta}\rangle\right] = \int_0^{\infty} ds \int \frac{d^3\vec{u}}{(2\pi)^2}\ \epsilon(p_0)\,\delta\big(p_0^2-(\vec{p}-\vec{u})^2-s\big)\,\widetilde{D}_{\beta}(\vec{u},s)$$

<u>Note</u>: this is a ***non-perturbative*** representation!

"Thermal spectral density"

- In the limit of vanishing temperature: $\widetilde{D}_{\beta}(\vec{u},s) \xrightarrow{\beta\to\infty} (2\pi)^3\delta^3(\vec{u})\,\rho(s)$

… and one recovers the standard Källén-Lehmann spectral representation

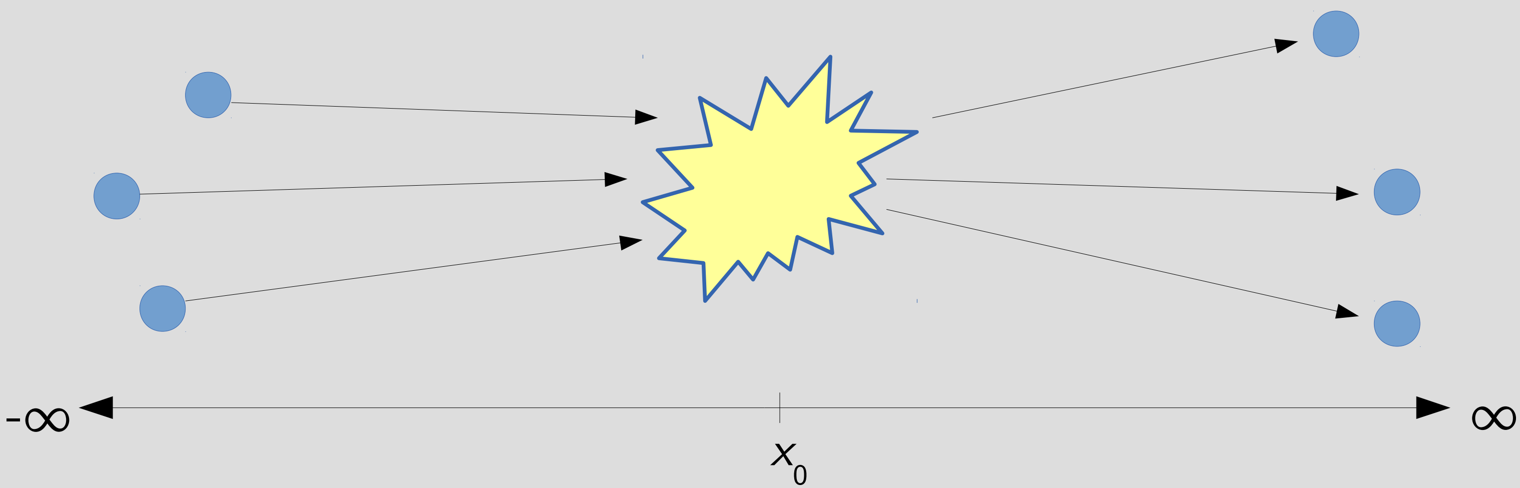
$$\widetilde{C}_{\beta}(p_0,\vec{p}) \xrightarrow{\beta\to\infty} 2\pi\,\epsilon(p_0)\int_0^{\infty} ds\ \delta\big(p^2-s\big)\,\rho(s)$$

e.g. $\rho(s)=\delta(s\text{-}m^2)$ for a massive free theory

4. Shear viscosity in Φ^4 theory

- Since all observable quantities are computed using correlation functions, one can use this spectral representation to gain new insights into the properties of QFTs at finite temperature

- As an example, in the recent work **2104.13413** this representation was used to calculate the shear viscosity η_0 arising from states at *large times* x_0 $\rightarrow$ *"thermal asymptotic states"*



Important: Interactions with the thermal background persist, even for large x_0

$\rightarrow$ ***Need to include an external coupling*** λ

4. Shear viscosity in Φ^4 theory

- A solution to the problem of asymptotic states in (scalar) thermal QFTs was proposed in hep-ph/0109136

→ *Asymptotic fields Φ_0 are assumed to satisfy dynamical equations only at large x_0*

$$(\partial^2 + m^2)\phi_0(x) + \frac{\lambda}{3!}\phi_0^3(x) \xrightarrow{|x_0|\to\infty} 0$$

- Given that the thermal spectral density has the decomposition:

$$\widetilde{D}_\beta(\vec{u}, s) = \widetilde{D}_{m,\beta}(\vec{u})\,\delta(s - m^2) + \widetilde{D}_{c,\beta}(\vec{u}, s)$$

"Thermal damping factor" → *Causes mass poles m to become screened by thermal effects*

Continuous component → *Suppressed at large x_0*

*… the thermal damping factor is **uniquely** fixed by the asymptotic field equation!*

- This means that the non-perturbative thermal effects experienced by particle states are entirely controlled by the asymptotic dynamics

4. Shear viscosity in Φ^4 theory

- Applying the asymptotic field condition for Φ^4 theory, the corresponding thermal damping factors have the form (see hep-ph/0109136)

$\rightarrow$ **For $\lambda < 0$** $$D_{m,\beta}(\vec{x}) = \frac{\sin(\kappa|\vec{x}|)}{\kappa|\vec{x}|}$$ $\rightarrow$ **For $\lambda > 0$** $$D_{m,\beta}(\vec{x}) = \frac{e^{-\kappa|\vec{x}|}}{\kappa_0|\vec{x}|}$$

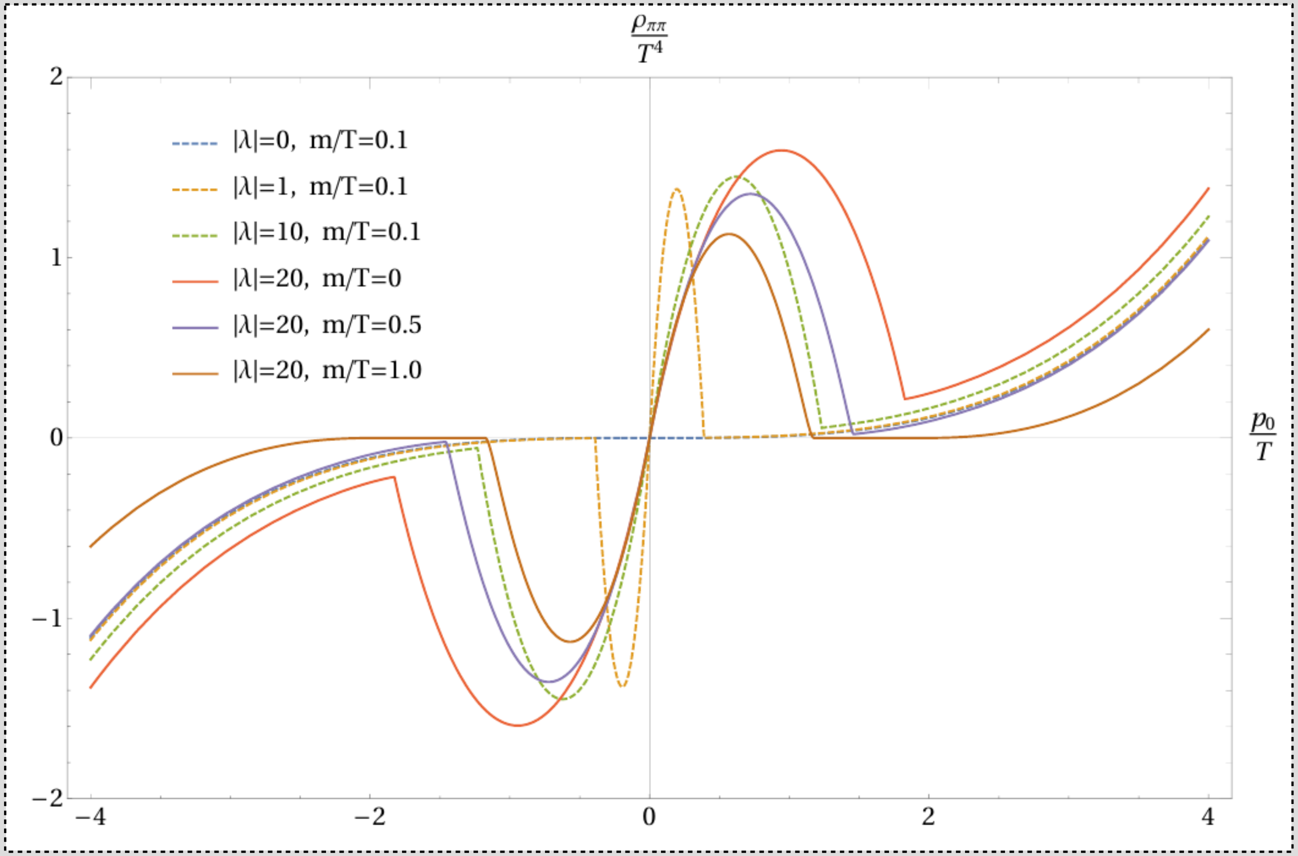
where κ is defined with $r = m/T$: $$\kappa = T\sqrt{|\lambda|}K(r), \qquad K(r) = \sqrt{\int \frac{d^3\hat{\vec{q}}}{(2\pi)^3 2\sqrt{|\hat{\vec{q}}|^2 + r^2}} \frac{1}{e^{\sqrt{|\hat{\vec{q}}|^2+r^2}} - 1}}$$

- Now that one has an explicit expression for the damping factors of the asymptotic states, one can use these to calculate the ***exact*** form of the EMT spectral function $$\rho_{\pi\pi}(p_0) = \lim_{\vec{p}\to 0} \mathcal{F}\left[\langle\Omega_\beta|\left[\pi^{ij}(x), \pi_{ij}(y)\right]|\Omega_\beta\rangle\right](p)$$

... the shear viscosity is then recovered via the Kubo relation $$\eta = \frac{1}{20}\lim_{p_0\to 0}\frac{d\rho_{\pi\pi}}{dp_0}$$

4. Shear viscosity in Φ^4 theory

- For $\lambda < 0$, the EMT spectral function $\rho_{\pi\pi}$ has the structure:



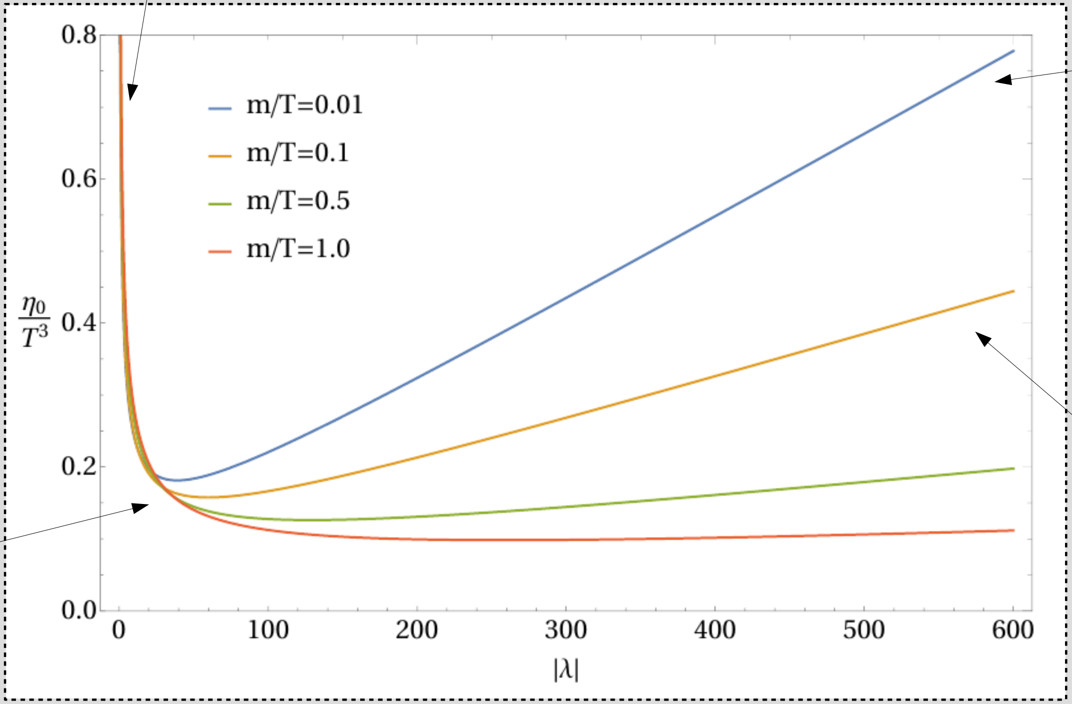
- The presence of interactions causes resonant peaks to appear $\rightarrow$ peaked when $p_0 \sim \kappa = 1/\ell$
- For $\lambda \rightarrow 0$ the free-field result is recovered, as expected
- The dimensionless ratio m/T controls the magnitude of the peaks

4. Shear viscosity in Φ^4 theory

- Applying the Kubo relation to $\rho_{\pi\pi}$ one ultimately obtains the following expression for the shear viscosity η_0

$$\eta_0 = \frac{T^3}{15\pi}\left[\frac{\mathcal{K}_3\left(\frac{m}{T},0,\infty\right)}{\sqrt{|\lambda|}} + \sqrt{|\lambda|}\,\mathcal{K}_1\left(\frac{m}{T},0,\infty\right) + \frac{\mathcal{K}_4\left(\frac{m}{T},\sqrt{|\lambda|}K\left(\frac{m}{T}\right),\sqrt{|\lambda|}K\left(\frac{m}{T}\right)\right)}{4|\lambda|}\right]$$

Dominant component for small $|\lambda|$



For large $|\boldsymbol{\lambda}|$, $\boldsymbol{\eta_0} \sim |\boldsymbol{\lambda}|$

Global minima

Magnitude of large $|\boldsymbol{\lambda}|$ growth controlled by $\boldsymbol{m/T}$

→ For fixed coupling, η_0/T^3 is entirely controlled by functions of m/T

4. Shear viscosity in Φ^4 theory

- What about $\lambda > 0$? $\rightarrow$ η_0 *diverges!*

  Why? – The thermal damping factor $D_{m,\beta}(\mathbf{u})$ does not decay rapidly enough at large momenta $\rightarrow$ *UV behaviour of the quartic interaction*

- For a thermal scalar QFT, in general:

  If the KMS condition holds $\Longrightarrow$ $D_{\beta}(\mathbf{u},s) \sim e^{-\beta|\mathbf{u}|/2}$, for $|\mathbf{u}|\rightarrow\infty$

  $\rightarrow$ This implies that when $\lambda>0$, the KMS condition does not hold in Φ^4 theory, i.e. *thermal equilibrium is violated!*

- In 2104.13413 these analytic conditions were used to relate the boundedness of η_0 with the existence of thermal equilibrium:

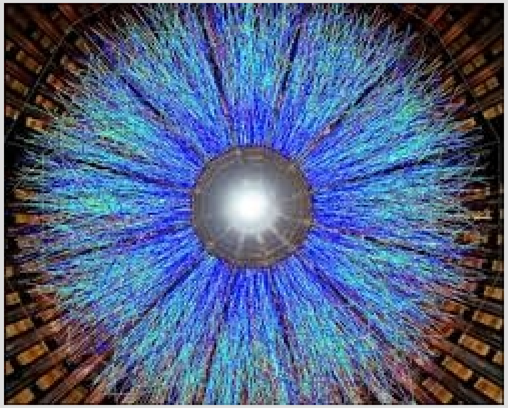
  If the KMS condition holds $\Longrightarrow$ η_0 is finite

5. Summary & outlook

- Local QFT is an analytic framework that attempts to address the fundamental question "*what is a QFT?*"
- The framework can be extended to $T > 0$ $\rightarrow$ This has important implications, including the generalisation of the Källén-Lehmann representation
- In 2104.13413, this representation was used to calculate the shear viscosity arising from asymptotic states η_0, a *non-perturbative* quantity
- So far, only real scalar fields $\Phi(x)$ were considered, where $T > 0$

  $\rightarrow$ In principle, this approach can be extended to non-scalar fields, as well as theories with $\mu \neq 0$ (work in progress!)

- The generalised KL representation could also enable
  - The extraction of observables from *Euclidean* data
  - New insights into the phase structure of QFTs



[Brookhaven National Lab]

Backup

- For thermal asymptotic states, the spectral function $\rho_{\pi\pi}$ has the form

$$\rho_{\pi\pi}(p_0) = \sinh\left(\frac{\beta}{2}p_0\right) \int \frac{d^3\vec{q}}{(2\pi)^4}\frac{2}{3}|\vec{q}|^4 \int_{-\infty}^{\infty} dq_0 \, \frac{\widetilde{C}_\beta(q_0,\vec{q})\,\widetilde{C}_\beta(p_0-q_0,\vec{q})}{\sinh\left(\frac{\beta}{2}q_0\right)\sinh\left(\frac{\beta}{2}(p_0-q_0)\right)}$$

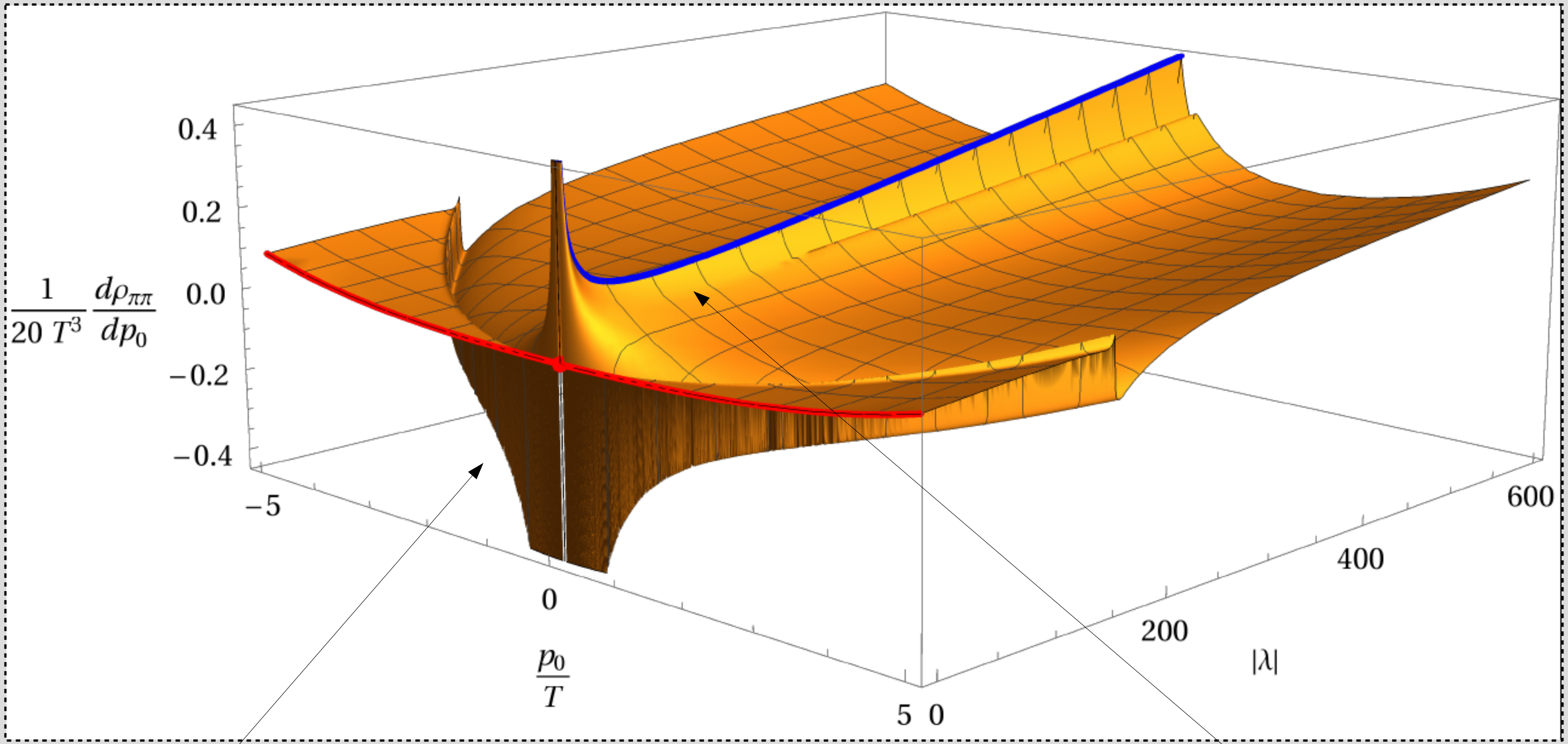
… which after applying the generalised KL representation, together with the Kubo relation, implies

$$\eta_0 = \frac{T^5}{240\pi^5}\int_0^\infty ds \int_0^\infty dt \int_0^\infty d|\vec{u}| \int_0^\infty d|\vec{v}| \, |\vec{u}||\vec{v}| \, \widetilde{D}_\beta(\vec{u},s)\,\widetilde{D}_\beta(\vec{v},t)$$
$$\times \left[4\left[1+\epsilon(|\vec{u}|-|\vec{v}|)\right]\left\{\frac{|\vec{v}|}{T}\mathcal{I}_3\left(\frac{\sqrt{t}}{T},0,\infty\right) + \frac{|\vec{v}|^3}{T^3}\mathcal{I}_1\left(\frac{\sqrt{t}}{T},0,\infty\right)\right\}\right.$$
$$\left. + \left\{\mathcal{I}_4\left(\frac{\sqrt{t}}{T},\frac{|\vec{v}|}{T},\frac{s-t+(|\vec{u}|+|\vec{v}|)^2}{2(|\vec{u}|+|\vec{v}|)T}\right) + \epsilon(|\vec{u}|-|\vec{v}|)\,\mathcal{I}_4\left(\frac{\sqrt{t}}{T},\frac{|\vec{v}|}{T},\frac{s-t+(|\vec{v}|-|\vec{u}|)^2}{2(|\vec{v}|-|\vec{u}|)T}\right)\right\}\right]$$

- The model dependence of η_0 factorises, and is controlled by the thermal spectral density $D_\beta(\mathbf{u},s)$

Backup

- For $\lambda < 0$, $\rho_{\pi\pi}(p_0)$ and its derivative are ***non-analytic*** at $(p_0/T, |\lambda|)=(0,0)$



Setting λ=0 first, and then $p_0/T \to 0$, leads to a *vanishing* result

But, setting $p_0/T=0$ first, and then $\lambda \to 0$, leads to a *divergent* result

→ η_0 in the interacting theory is not a continuous perturbation of the free field result ($\eta_0 = 0$)