# Quark Free Energies and Electric Fluxes

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Based On:
MG, von Smekal, arXiv:2206.11697

## Confinement/Deconfinement

- pure gauge theory:
  - first-order transition ($\mathbb{Z}_3$-symmetry):
    Polyakov loop $\langle L \rangle \iff$ infinitely heavy charge
  - 't Hooft string tension, static quark-anti-quark potential, etc.

## Free Energy (Static Quarks)

- free energy difference

$$\mathrm{e}^{-\frac{1}{T}\Delta F_{\mathrm{q}}} = \langle L \rangle$$

- confinement/deconfinement:

$$\Delta F_{\mathrm{q}} = \begin{cases} < \infty, & \text{deconfined} \\ \infty, & \text{confined} \end{cases}$$

## Free Energy (Dynamical Quarks)

- naively:

$$\Delta F_{\mathrm{q}} = -T \ln \frac{Z_{N=1}}{Z_{N=0}}$$

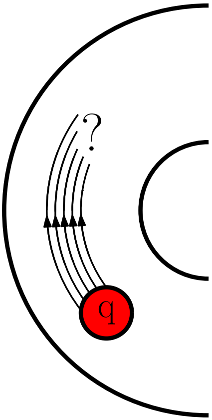
## Naive Approach

- fugacity expansion:

$$Z(T, \mu = \mathrm{i}\theta T) = \sum_N \mathrm{e}^{\mathrm{i}N\theta} Z_N(T)$$

- canonical ensemble:

$$Z_N(T) = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\theta \, \mathrm{e}^{-\mathrm{i}N\theta} Z(T, \mathrm{i}\theta T)$$




## Problem

- Roberge-Weiss-Symmetry:

$$Z(\theta) = Z\left(\theta + \frac{2\pi}{3}\right)$$

- $Z_N = 0$ for all $N \neq 0 \mod 3$

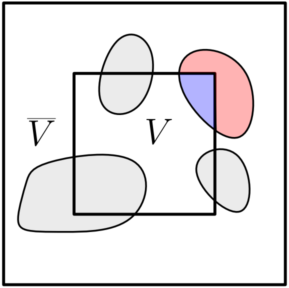
## Goal/Objective

Construct ensemble $Z_{N=1,2 \bmod 3}$ of fractional baryon number, e.g.

$$N = 1 \mod 3 = \ldots, -11, -8, -5, -2, 1, 4, 7, 10, \ldots$$

## Result

- construction for partial volume $V$ of lattice.
- Flux-Tube Model (easy)
- full QCD (difficult)



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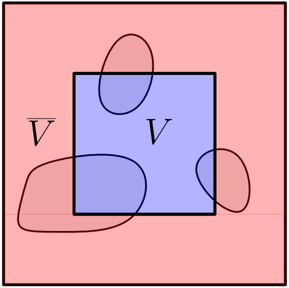


## Spacetime Geometry

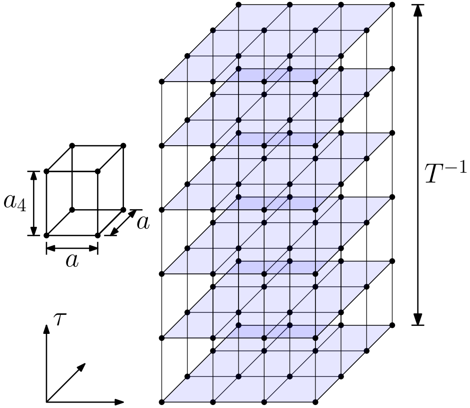
- hyper-cubical lattice
- in all directions:
  - finite ($L_\mu < \infty$)
  - periodic (torus)

## Thermodynamic Ensemble

- partition function:

$$Z(T,\mu) = \int \mathcal{D}[\dots]\, \mathrm{e}^{S}$$

- $T^{-1} = L_4 a_4$

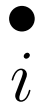



## Imaginary Time

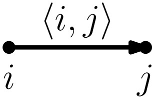
$$Z(T,\mu) = \mathrm{tr}(\mathrm{e}^{-\beta \hat{H}}) = \mathrm{tr}(\underbrace{\mathrm{e}^{-a_4 \hat{H}} \dots \mathrm{e}^{-a_4 \hat{H}}}_{L_4 \text{ times}})$$

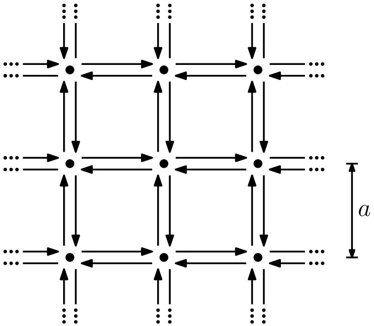
## Basic Objects

- sites:




- links:




## Field Variables

- fermions: $\overline{\psi}_i^{a\alpha}$ and $\psi_i^{a\alpha}$ (Grassmann numbers)
- gauge fields: $U_{\langle i,j\rangle} \in \mathrm{SU}(3)$ and $U_{\langle j,i\rangle} = U^\dagger_{\langle i,j\rangle}$

## Lattice Action

$$S(\overline{\psi}, \psi, U) = S_{\mathrm{Gauge}}(U) + S_{\mathrm{Fermion}}(\overline{\psi}, \psi, U)$$

## How is $S$ chosen?

- gauge-invariance ($\Omega_i \in \mathrm{SU}(3)$)

$$\psi_i \longrightarrow \Omega_i \psi_i, \quad \overline{\psi}_i \longrightarrow \overline{\psi}_i \Omega_i^\dagger, \quad U_{\langle i,j\rangle} \longrightarrow \Omega_i U_{\langle i,j\rangle} \Omega_j^\dagger$$

- require naive limit:

$$S(\overline{\psi}, \psi, U) \longrightarrow S_{\mathrm{QCD}}(\overline{\psi}(x), \psi(x), A_\mu(x)) \qquad (a_4, a \to 0^+)$$

- This is NOT rigorous! Merely a HEURISTIC guess!

## Here

Wilson Plaquette Action + Wilson Fermions (one flavor)

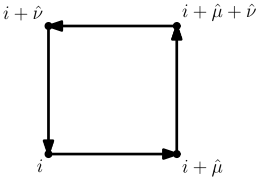
# Wilson Plaquette Action

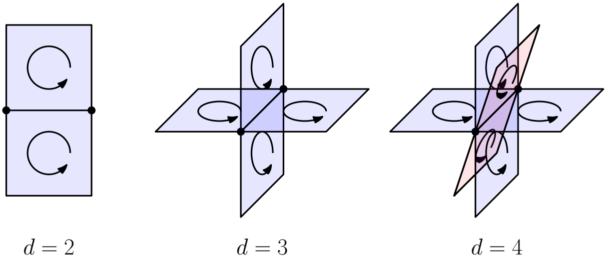
- plaquette:

$$U_P = U_{\langle i,i+\hat{\mu}\rangle} U_{\langle i+\hat{\mu},i+\hat{\mu}+\hat{\nu}\rangle} U_{\langle i+\hat{\mu}+\hat{\nu},i+\hat{\nu}\rangle} U_{\langle i+\hat{\nu},i\rangle}$$

- action:

$$S_{\text{Gauge}}(U) = \frac{2}{g_0^2} \sum_P \mathrm{Re}\mathrm{Tr}\, U_P$$



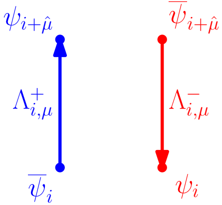



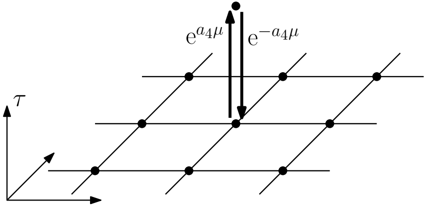
## Wilson Fermions

$$S_{\text{Fermion}}(\overline{\psi}, \psi, U) = \sum_{i,\mu} \kappa \left[ \overline{\psi}_i \Lambda^+_{i,\mu} \psi_{i+\hat{\mu}} + \overline{\psi}_{i+\hat{\mu}} \Lambda^-_{i,\mu} \psi_i \right] - \sum_i \overline{\psi}_i \psi_i$$

with

$$\Lambda^+_{i,\mu} = ((1 - \delta_{\mu,4}) + \mathrm{e}^{+a_4\mu}\delta_{\mu,4})(1 - \gamma_\mu)U_{\langle i,i+\hat{\mu}\rangle}$$
$$\Lambda^-_{i,\mu} = ((1 - \delta_{\mu,4}) + \mathrm{e}^{-a_4\mu}\delta_{\mu,4})(1 + \gamma_\mu)U_{\langle i+\hat{\mu},i\rangle}$$



## Partition Function

$$
\begin{aligned}
Z(T,\mu) &= \int \mathcal{D}U\mathcal{D}\overline{\psi}\mathcal{D}\psi \, \mathrm{e}^{S_{\text{Gauge}}+S_{\text{Fermion}}} \\
&= \int \mathcal{D}U \, \mathrm{e}^{S_{\text{Gauge}}} \left( \int \mathcal{D}\overline{\psi}\mathcal{D}\psi \, \mathrm{e}^{S_{\text{Fermion}}} \right) \\
&= \int \mathcal{D}U \, \mathrm{e}^{S_{\text{Gauge}}(U)} \quad \underbrace{\det M(U)}_{\text{fermion determinant}}
\end{aligned}
$$

## Continuum Limit

- define continuum limit $a_4, a \to 0^+$ such that
  - whatever comes out resembles a relativistic QFT
  - physical quantities are reproduced correctly (e.g. hadron masses,...)

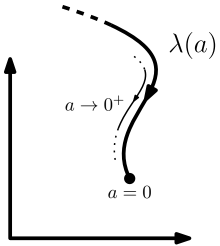
## Scaling Limit

Assume $a_4 = a$.

$$R_a(g_0, \kappa, a\mu) = (\mathcal{O}_1, \mathcal{O}_2, \ldots)$$

Line of constant physics:

$$\lambda(a) = R_a^{-1}(\mathcal{O}_1, \mathcal{O}_2, \ldots)$$



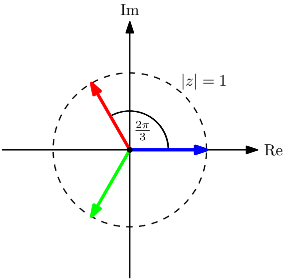
## Effective Theory (Requirements)

- only spatial lattice
- local quark numbers
- Roberge-Weiss-Symmetry
- local Gauss Law (charges and fluxes)

## Idea

Reduce all degrees of freedom to center elements only:

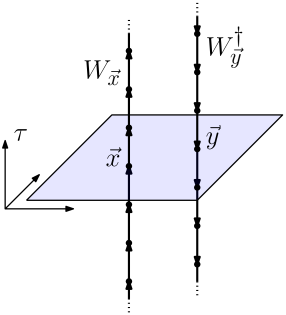
$$Z_3 = \left\{ 1, \, e^{i\frac{2\pi}{3}}, e^{i\frac{4\pi}{3}} \right\} \leq SU(3)$$



## Polyakov Loop

$$W_{\vec{x}} = \prod_{\langle i,j \rangle \in \mathcal{C}_{\vec{x}}} U_{\langle i,j \rangle} \in \mathrm{SU}(3)$$

$$L_{\vec{x}} = \mathrm{tr}\, W_{\vec{x}}$$




## Effective Theory

$$Z = \int \mathcal{D}U \, \mathrm{e}^{S_{\mathrm{Gauge}}(U)} \det M(U) \longrightarrow \int \mathcal{D}L \, \mathrm{e}^{\tilde{S}_{\mathrm{Gauge}}(L)} \prod_{\vec{x}} Q(L_{\vec{x}})$$

$$\longrightarrow \frac{1}{3^{N_s}} \sum_{\{z_{\vec{x}}\}} \mathrm{e}^{S_{\mathrm{eff}}(z)} \prod_{\vec{x}} Q(z_{\vec{x}})$$

## Strong Coupling and Heavy Mass Limit[1]

- effective action

$$Z = \int \mathcal{D}U_4 \int \mathcal{D}U_k \, \mathrm{e}^{S_{\text{Gauge}}} \det M = \int \mathcal{D}U_4 \, \mathrm{e}^{S_{\text{eff}}}$$

- expand $S_{\text{eff}}$ around $g_0 \to \infty$ (strong coupling), $\kappa = 0$ (heavy mass):

$$\mathrm{e}^{S_{\text{eff}}} \approx \left( \prod_{\langle \vec{x}, \vec{y} \rangle} \left[ 1 + 2\lambda \mathrm{Re}\, L_{\vec{x}} L^*_{\vec{y}} \right] \right) \prod_{\vec{x}} Q(L_{\vec{x}})$$

with

$$Q(L_{\vec{x}}) = \left(1 + hL_{\vec{x}} + h^2 L^*_{\vec{x}} + h^3\right)^2 \left(1 + \overline{h} L^*_{\vec{x}} + \overline{h}^2 L_{\vec{x}} + \overline{h}^3\right)^2$$

$$h = (2\kappa \mathrm{e}^{a_4 \mu})^{L_4} = (\mathrm{e}^{-a_4 m} \mathrm{e}^{a_4 \mu})^{L_4} = \mathrm{e}^{(\mu - m)/T}$$

$$\overline{h} = (2\kappa \mathrm{e}^{-a_4 \mu})^{L_4} = (\mathrm{e}^{-a_4 m} \mathrm{e}^{-a_4 \mu})^{L_4} = \mathrm{e}^{-(\mu + m)/T}$$

[1] J. Langelage, M. Neuman, O. Philipsen, S. Lottini, M. Fromm (JHEP 2011, 57 (2011); 2012, 42 (2012); 2014, 131 (2014))

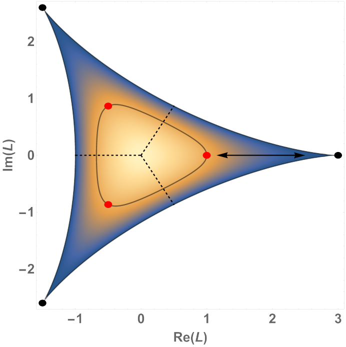
## Approximation of Group Integration

$$\int \mathrm{d}U\, f(\mathrm{tr}(U)) \longrightarrow \frac{1}{3}\sum_{z\in \mathrm{Z}_3} f(3z)$$

$$\int \mathrm{d}L\, J(L) f(L) \longrightarrow \frac{1}{3}\sum_{z\in \mathrm{Z}_3} f(z)$$

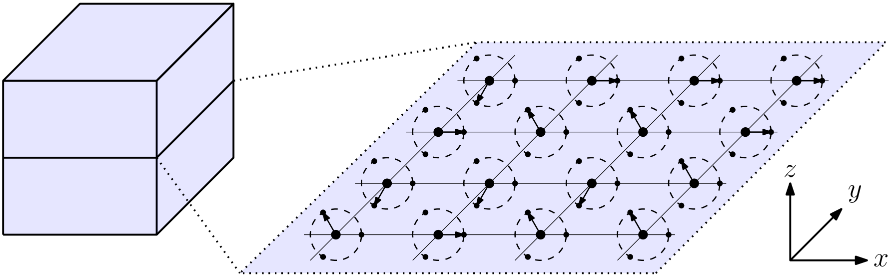
- maintain invariance:

$$\frac{1}{3}\sum_{z\in \mathrm{Z}_3} f(w\cdot z) = \frac{1}{3}\sum_{z\in \mathrm{Z}_3} f(z)$$



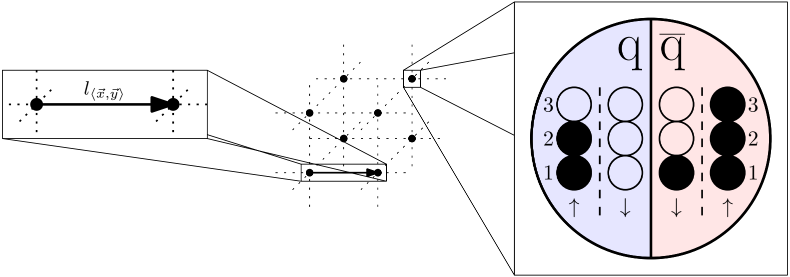
## Effective Theory

$$Z_{\text{eff}}(T,\mu) = \frac{1}{3^{N_s}} \sum_{\{z_{\vec{x}}\}} \left( \prod_{\langle \vec{x},\vec{y} \rangle} \left[ 1 + 2\lambda \text{Re}\, z_{\vec{x}} z_{\vec{y}}^* \right] \right) \prod_{\vec{x}} Q(z_{\vec{x}})$$



## Flux-Tube Model[1]

- flux tube (electric fluxes): $l_{\langle\vec{x},\vec{y}\rangle} \in \{1,0,-1\}$ $(l_{\langle\vec{y},\vec{x}\rangle} = -l_{\langle\vec{x},\vec{y}\rangle})$
- quark and anti-quarks: $\underbrace{n_{\vec{x},\uparrow}, n_{\vec{x},\downarrow}}_{\text{quarks}}, \underbrace{\overline{n}_{\vec{x},\uparrow}, \overline{n}_{\vec{x},\downarrow}}_{\text{anti}-\text{quarks}} \in \{0,1,2,3\}$



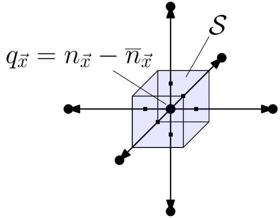

[1] A. Patel, Nucl. Phys. B243 (1984) 411, C. Bernard et al., Phys. Rev. D 49 (1994), 6051, J. Condella and C. DeTar, Phys. Rev. D 61 (2000), 074023

## Hamiltonian

$$H(\{l,n\}) = \sum_{\langle\vec{x},\vec{y}\rangle} \sigma a |l_{\langle\vec{x},\vec{y}\rangle}| + \sum_{\vec{x},s=\uparrow,\downarrow} m(n_{\vec{x},s} + \overline{n}_{\vec{x},s})$$

## Local Gauss Law

$$q_{\vec{x}} = \underbrace{\sum_{\vec{y}\sim\vec{x}} l_{\langle\vec{x},\vec{y}\rangle}}_{=\phi_{\mathcal{S}}=\phi_{\vec{x}}} \mod 3$$



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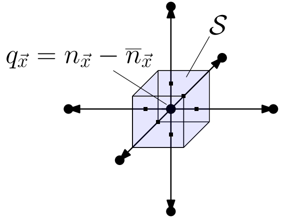



## Flux-Tube Model

$$Z_{\text{flux}}(T,\mu) = \sum_{\{l,n\}} \mathrm{e}^{-\frac{1}{T}H(\{l,n\}) + \frac{\mu}{T}\sum_{\vec{x}} q_{\vec{x}}} \prod_{\vec{x}} \delta(q_{\vec{x}} = \phi_{\vec{x}} \mod 3)$$

## Flux-Tube Model $\equiv$ Effective Theory

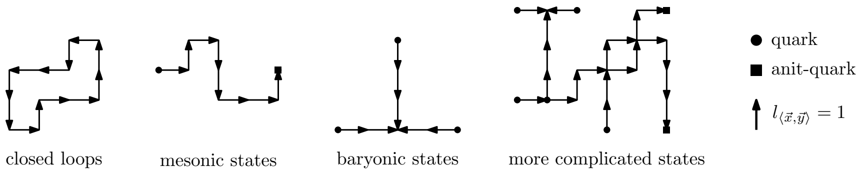
$$
\begin{aligned}
Z_{\text{flux}}(T,\mu) &= \sum_{\{l,n\}} \mathrm{e}^{-\frac{1}{T}H(\{l,n\})+\frac{\mu}{T}\sum_{\vec{x}} q_{\vec{x}}} \prod_{\vec{x}} \delta(q_{\vec{x}} = \phi_{\vec{x}} \mod 3) \\
&= \sum_{\{l,n\}} \mathrm{e}^{-\frac{1}{T}H(\{l,n\})+\frac{\mu}{T}\sum_{\vec{x}} q_{\vec{x}}} \prod_{\vec{x}} \frac{1}{3} \sum_{z\in \mathrm{Z}_3} z^{q_{\vec{x}}-\phi_{\vec{x}}} \\
&= \frac{1}{3^{N_s}} \sum_{\{z_{\vec{x}}\}} \dots \\
&= \frac{1}{3^{N_s}} \sum_{\{z_{\vec{x}}\}} \left( \prod_{\langle \vec{x},\vec{y}\rangle} \left[1 + 2\mathrm{e}^{-\beta\sigma a} \mathrm{Re}\, z_{\vec{x}} z_{\vec{y}}^{*}\right]\right) \prod_{\vec{x}} Q(z_{\vec{x}}) \\
&= Z_{\text{eff}}(T,\mu), \quad \lambda = \mathrm{e}^{-\beta\sigma a}
\end{aligned}
$$

## Total Charge

$$\sum_{\vec{x}} q_{\vec{x}} \mod 3 = \sum_{\vec{x}} \overbrace{\sum_{\vec{y}\sim\vec{x}} l_{\langle\vec{x},\vec{y}\rangle}}^{=\phi_{\vec{x}}} = \sum_{\langle\vec{x},\vec{y}\rangle} (l_{\langle\vec{x},\vec{y}\rangle} + l_{\langle\vec{y},\vec{x}\rangle})$$

$$= \sum_{\langle\vec{x},\vec{y}\rangle} (l_{\langle\vec{x},\vec{y}\rangle} - l_{\langle\vec{x},\vec{y}\rangle})$$

$$= 0$$



## Total Charge

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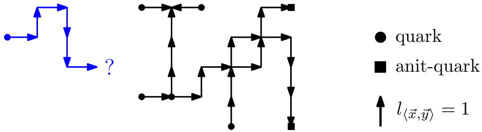
\begin{aligned}
\sum_{\vec{x}} q_{\vec{x}} \mod 3 = \sum_{\vec{x}} \overbrace{\sum_{\vec{y}\sim\vec{x}} l_{\langle\vec{x},\vec{y}\rangle}}^{=\phi_{\vec{x}}} &= \sum_{\langle\vec{x},\vec{y}\rangle} \left(l_{\langle\vec{x},\vec{y}\rangle} + l_{\langle\vec{y},\vec{x}\rangle}\right) \\
&= \sum_{\langle\vec{x},\vec{y}\rangle} \left(l_{\langle\vec{x},\vec{y}\rangle} - l_{\langle\vec{x},\vec{y}\rangle}\right) \\
&= 0
\end{aligned}
$$



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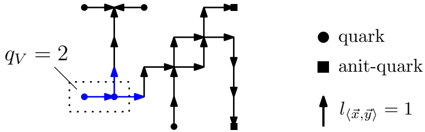
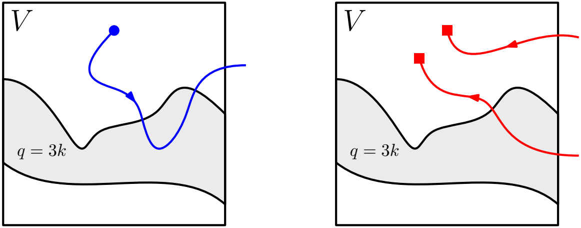


## Ensemble with Fixed Quark Number

$$Z^{(1)}_{\text{flux}}(T,\mu) = \sum_{\{l,n\}} \ldots \delta(q_V = 1 \mod 3) \prod_{\vec{x}} \delta(q_{\vec{x}} = \phi_{\vec{x}} \mod 3)$$

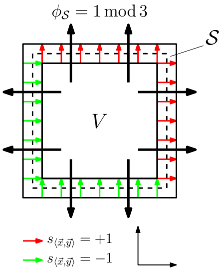
- fix net quark number in partial volume $V$, e.g. $q_V = 1 \mod 3$:

$$q_V = \underbrace{\ldots, -11, -8, -5, -2,}_{\text{two additional anti-quarks}} \quad \underbrace{1, 4, 7, 10, \ldots}_{\text{one additional quark}}$$



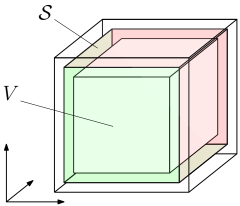
## Flux-Tube Model $\equiv$ Effective Theory

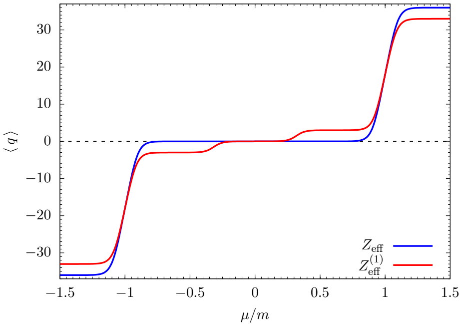
$$Z_{\text{flux}}^{(1)} = \ldots = \frac{1}{3}\sum_{k=0}^{2} \mathrm{e}^{-\mathrm{i}\frac{2\pi}{3}k} Z_{\text{eff}}(k) = Z_{\text{eff}}^{(1)}$$




## Twisted Ensemble

$$Z_{\text{eff}}(k) = \prod_{\langle\vec{x},\vec{y}\rangle} \left(1 + 2\lambda \mathrm{Re}\left[\mathrm{e}^{-\mathrm{i}\frac{2\pi}{3} s_{\langle\vec{x},\vec{y}\rangle} k} z_{\vec{x}} z_{\vec{y}}^*\right]\right) \times \prod_{\vec{x}} Q(z_{\vec{x}})$$





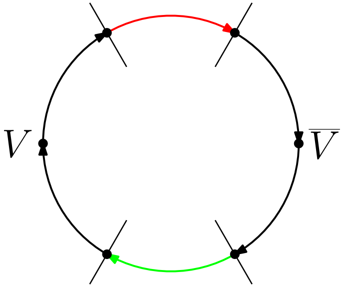
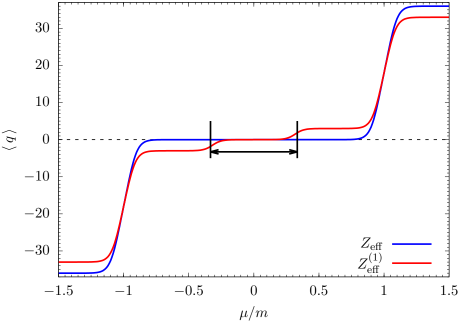




Figure: $\sigma a/m = 0.3$ and $T/m = 0.1$



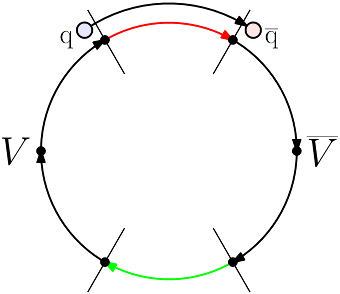
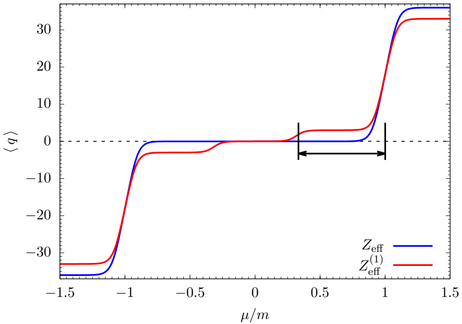




Figure: $\sigma a/m = 0.3$ and $T/m = 0.1$



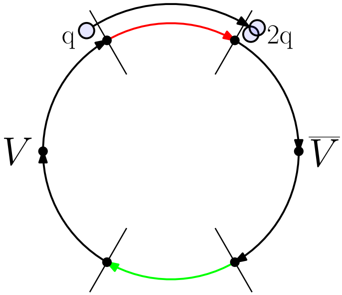
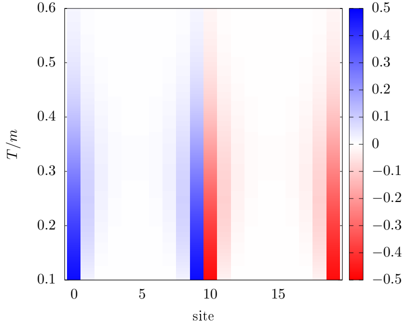
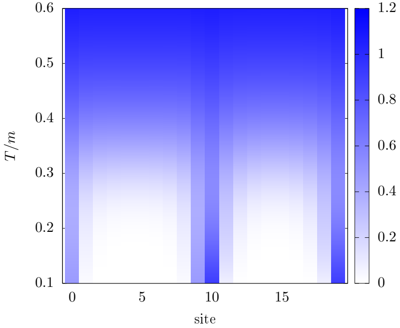




Figure: $\sigma a/m = 0.3$ and $T/m = 0.1$




(a) $\mu/m = 0$ (mesonic)




(b) $\mu/m = 1/2$ (baryonic)

Figure: $L = 20$, $\sigma a/m = 0.3$

## Free Energy

$$\Delta F_\infty = \lim_{L\to\infty}\left(-T\ln\frac{Z_{\text{eff}}^{(1)}}{Z_{\text{eff}}^{(0)}}\right)$$

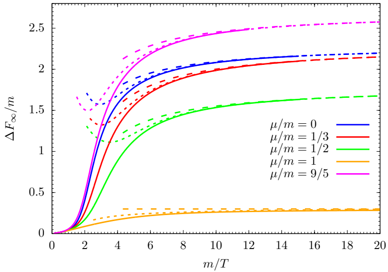



Figure: $L = 64$, $\sigma a/m = 0.3$

## Low Temperature Expansion

$$\Delta F = \left(\Delta_0^{(1)} - \Delta_0^{(0)}\right) - T\ln\left(\frac{N_0^{(1)} + N_1^{(1)}\mathrm{e}^{-(\Delta_1^{(1)}-\Delta_0^{(1)})/T} + \dots}{N_0^{(0)} + N_1^{(0)}\mathrm{e}^{-(\Delta_1^{(0)}-\Delta_0^{(0)})/T} + \dots}\right)$$

$$\Delta = H - \mu q$$

## Free Energy

$$\Delta F_\infty = \lim_{L\to\infty}\left(-T\ln\frac{Z_{\text{eff}}^{(1)}}{Z_{\text{eff}}^{(0)}}\right)$$

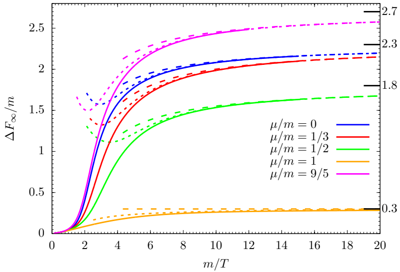



Figure: $L = 64$, $\sigma a/m = 0.3$

## Low Temperature Expansion

$$\Delta_0^{(1)} - \Delta_0^{(0)} = \begin{cases} 2m + \sigma a, & \mu \leq m/3 \\ 3|m-\mu| + \sigma a, & \mu > m/3 \end{cases}$$

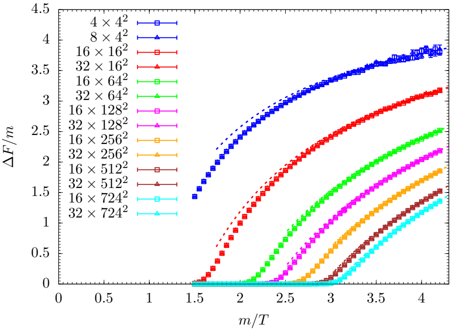
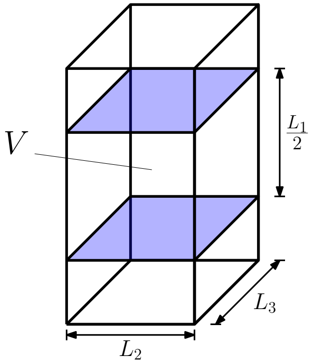



Figure: $\sigma a/m = 3$, $\mu/m = 0$




$L_1 \times (L_2 L_3)$

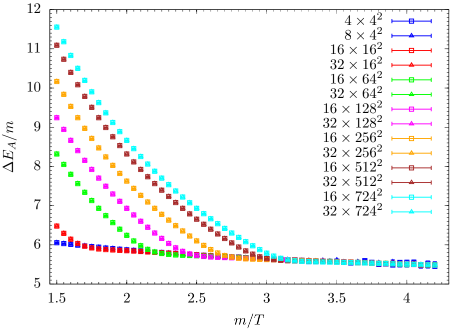
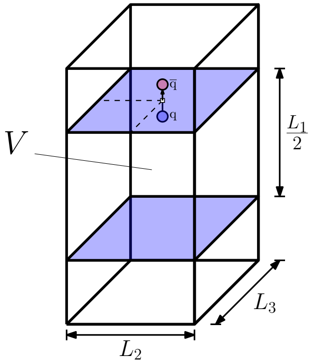



Figure: $\sigma a/m = 3$, $\mu/m = 0$




$L_1 \times (L_2 L_3)$

## Entropic Term

$$\Delta E_A = \Delta F - T \ln \Big( \underbrace{2}_{\text{interfaces}} \cdot \underbrace{4}_{\text{degeneracy}} \cdot (L_2 L_3) \Big)$$

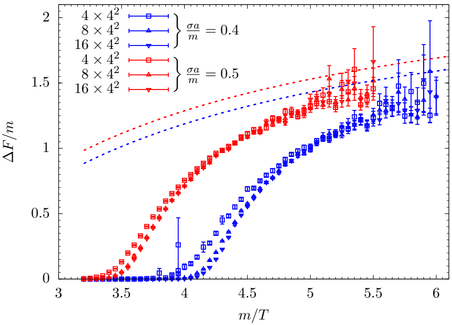
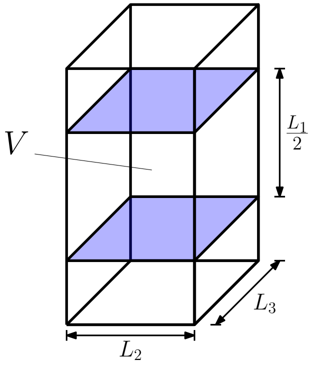
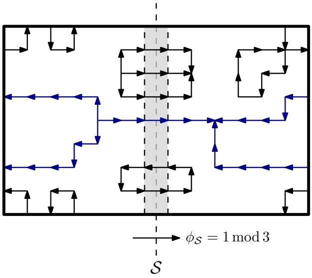



Figure: $\mu/m = 0$

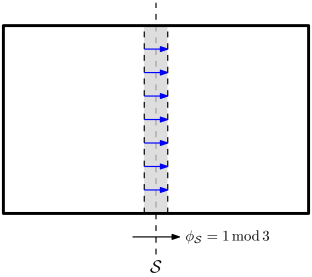



$L_1 \times (L_2 L_3)$

# Effective Theory



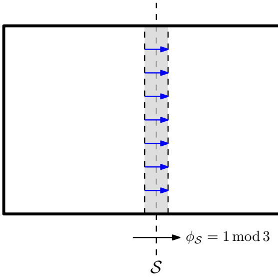
## Effective Theory




$$\mathrm{Re}\,\left(z_{\vec{x}} z^*_{\vec{y}}\right) \to \mathrm{Re}\,\left(\mathrm{e}^{-\mathrm{i}\frac{2\pi}{3}k} z_{\vec{x}} z^*_{\vec{y}}\right)$$

$$Z_{\mathrm{eff}}^{\phi_{\mathcal{S}},1} = \frac{1}{3}\sum_{k=0}^{2} \mathrm{e}^{-\mathrm{i}\frac{2\pi}{3}k} Z_{\mathrm{eff}}(k)$$

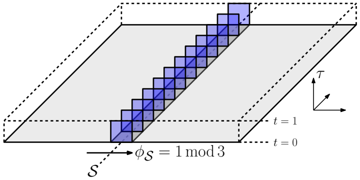
## Effective Theory




$$\mathrm{Re}\,\left(z_{\vec{x}} z^*_{\vec{y}}\right) \to \mathrm{Re}\,\left(\mathrm{e}^{-\mathrm{i}\frac{2\pi}{3}k} z_{\vec{x}} z^*_{\vec{y}}\right)$$

$$Z_{\mathrm{eff}}^{\phi_{\mathcal{S}},1} = \frac{1}{3}\sum_{k=0}^{2} \mathrm{e}^{-\mathrm{i}\frac{2\pi}{3}k} Z_{\mathrm{eff}}(k)$$

## Pure Gauge Theory[1]

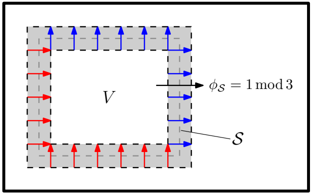



$$\mathrm{ReTr}\,\left(U_P\right) \to \mathrm{ReTr}\,\left(\mathrm{e}^{-\mathrm{i}\frac{2\pi}{3}k} U_P\right)$$

$$Z_{\phi_{\mathcal{S}},1} = \frac{1}{3}\sum_{k=0}^{2} \mathrm{e}^{-\mathrm{i}\frac{2\pi}{3}k} Z(k)$$

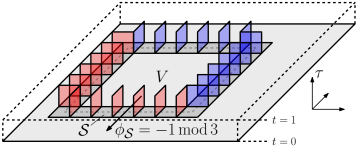
[1] 't Hooft, Nucl. Phys. B153 (1979) 141, L. G. Yaffe, Phys. Rev. D 21 (1980) 6, C. Borgs and E.Seiler, Commun.Math. Phys. 91 (1983) 329, P. de Forcrand and L. von Smekal, Phys. Rev. D 66 (2002) 1

## Effective Theory




$$\mathrm{Re}\left(z_{\vec{x}} z^*_{\vec{y}}\right) \to \begin{cases} \mathrm{Re}\left(\mathrm{e}^{-\mathrm{i}\frac{2\pi}{3}k} z_{\vec{x}} z^*_{\vec{y}}\right) \\ \mathrm{Re}\left(\mathrm{e}^{+\mathrm{i}\frac{2\pi}{3}k} z_{\vec{x}} z^*_{\vec{y}}\right) \end{cases}$$

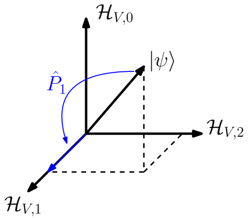
## Full QCD (Naive)




$$\mathrm{ReTr}\left(U_P\right) \to \begin{cases} \mathrm{ReTr}\left(\mathrm{e}^{+\mathrm{i}\frac{2\pi}{3}k} U_P\right) \\ \mathrm{ReTr}\left(\mathrm{e}^{-\mathrm{i}\frac{2\pi}{3}k} U_P\right) \end{cases}$$

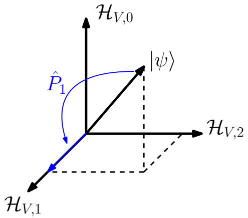
## Full QCD (Naive)

$$Z^{(1)} = \mathrm{tr}_{\mathcal{H}_{V,1}} \left( \mathrm{e}^{\frac{\mu}{T}\hat{N}} \mathrm{e}^{-\frac{1}{T}\hat{H}} \right) = \mathrm{tr} \left( \hat{P}_1 \mathrm{e}^{\frac{\mu}{T}\hat{N}} \mathrm{e}^{-\frac{1}{T}\hat{H}} \right)$$



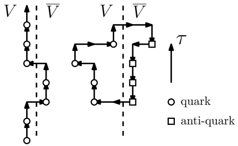
## Full QCD (Naive)

$$Z^{(1)} = \mathrm{tr}_{\mathcal{H}_{V,1}}\left(\mathrm{e}^{\frac{\mu}{T}\hat{N}}\mathrm{e}^{-\frac{1}{T}\hat{H}}\right) = \mathrm{tr}\left(\hat{P}_1\mathrm{e}^{\frac{\mu}{T}\hat{N}}\mathrm{e}^{-\frac{1}{T}\hat{H}}\right)$$



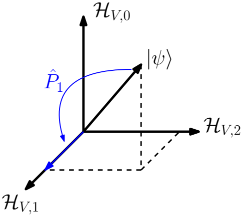

## Problem

$$[\hat{P}_1, \hat{H}] \neq 0$$



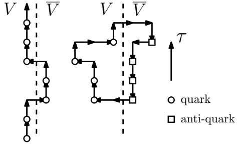
## Full QCD (Naive)

$$Z^{(1)} = \mathrm{tr}_{\mathcal{H}_{V,1}} \left( \mathrm{e}^{\frac{\mu}{T}\hat{N}} \mathrm{e}^{-\frac{1}{T}\hat{H}} \right) = \mathrm{tr} \left( \hat{P}_1 \mathrm{e}^{\frac{\mu}{T}\hat{N}} \mathrm{e}^{-\frac{1}{T}\hat{H}} \right)$$




## Problem

$$[\hat{P}_1, \hat{H}] \neq 0$$




## Solution

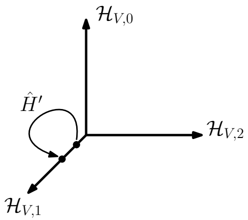
Modify $\hat{H}$ at surface $\mathcal{S}$:

$$\hat{H} \longrightarrow \hat{H}'$$

## Requirements

$\hat{H}'$ should be

- self-adjoint
- gauge-invariant
- derived from $\hat{H}$ (only modify at $\mathcal{S}$)
- $[\hat{P}_{0,1,2}, \hat{H}'] = 0$




## Modified Dynamics

$$\hat{H}' = \frac{1}{3} \sum_{z \in \mathrm{Z}_3} \hat{\phi}^z_{\mathcal{S}} \hat{H} \hat{\phi}^{z^{-1}}_{\mathcal{S}}$$

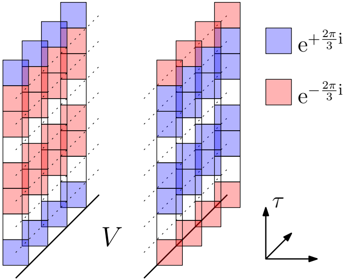
## Modified Dynamics

$$Z^{(1)} = \mathrm{tr}\left(\hat{P}_1 \mathrm{e}^{\frac{\mu}{T}\hat{N}} \mathrm{e}^{-\frac{1}{T}\hat{H}'}\right)$$

## Path Integral

$$Z^{(1)} = \frac{1}{3^{L_4}} \sum_{\{z\}} \left[\prod_{\tau=0}^{L_4-1} z_\tau^{-1}\right] Z(\{z\})$$

$$Z(\{z\}) = \int \mathcal{D}U \, \mathrm{e}^{S_{\mathrm{Gauge}}(\{z\})} \det M$$



# Thank You!